

ESSENTIALLY NORMAL COMPOSITION OPERATORS ON H^2

MOR KATZ

ABSTRACT. We prove a simple criterion for essential normality of composition operators on the Hardy space induced by maps in a reasonably large class \mathcal{S} of analytic self-maps of the unit disk. By combining this criterion with boundary Carathéodory-Fejér interpolation theory, we exhibit a parametrization for all rational self-maps of the unit disk which induce essentially normal composition operators.

1. INTRODUCTION

For φ an analytic self-map of the unit disk \mathbb{D} , the composition operator $C_\varphi: f \rightarrow f \circ \varphi$ induced by φ is a bounded operator on the Hardy space H^2 . A bounded operator A is said to be essentially normal if its self-commutator $[A^*, A] = A^*A - AA^*$ is a compact operator, and trivially essentially normal if A is either normal ($[A^*, A] = 0$) or compact. Normal composition operators on H^2 were characterized by Schwartz [26] and compact composition operators by Shapiro [27] and, via a different criterion, by Sarason [24] and Shapiro-Sundberg [28]; see also Cima-Matheson [10]. In [7], Bourdon-Levi-Narayan-Shapiro characterize the class of linear fractional self-maps of \mathbb{D} that induce non-trivially essentially normal composition operators; these maps are exactly the parabolic non-automorphisms of \mathbb{D} . These authors provide additional examples of essentially normal composition operators induced by maps which, like linear fractional non-automorphisms of \mathbb{D} , have order of contact 2 with $\partial\mathbb{D}$ at one point. To the best of the author's knowledge, no non-trivially essentially normal composition operators with inducing maps having order of contact $n > 2$ with $\partial\mathbb{D}$ were known prior to the present work.

In this paper we prove a simple criterion (Theorem 6.7 below) for essential normality of composition operators induced by maps φ in the class \mathcal{S} introduced by Kriete-Moorhouse in [21]. Roughly speaking, the class \mathcal{S} consists of analytic self-maps φ of \mathbb{D} that have “significant contact” with $\partial\mathbb{D}$ at only a finite number of points, with φ having “sufficient derivative data” at every such point. As a corollary, we show that for a self-map φ of \mathbb{D} which extends analytically to a neighborhood of $\overline{\mathbb{D}}$, C_φ is non-trivially essentially normal if and only if φ fixes one point of $\partial\mathbb{D}$, has derivative equal to 1 there, and maps the rest of $\partial\mathbb{D}$ into \mathbb{D} . Note that for the linear fractional case this criterion is equivalent to the characterization described above. Our criterion, in conjunction with boundary Carathéodory-Fejér interpolation theory, yields a parametrization for all rational self-maps φ of \mathbb{D} that induce non-trivially essentially normal C_φ on H^2 .

Date: March 17, 2015.

2000 Mathematics Subject Classification. Primary 47B33.

Keywords: composition operator, essentially normal, Hardy space.

We rely on results from three distinct areas, presented in Sections 3-5. First, we explore a special case of a boundary version of the Carathéodory-Fejér problem studied by Agler-Lykova-Young in [1, 2]. Second, we discuss relations in the Calkin Algebra using results by Kriete-Moorhouse [21]. In particular, we derive a decomposition of a composition operator modulo the ideal \mathcal{K} of compact operators into a sum of composition operators induced by “basic” rational functions (Theorem 4.4). Third, using formulas and ideas from Bourdon-Shapiro [8], based on work of Cowen-Gallardo [14] and Hammond-Moorhouse-Robbins [17], we obtain an operator formula for $C_\psi C_\varphi^*$ where φ is rational and ψ is an auxiliary map, and reduce this formula modulo \mathcal{K} . Additionally, Faà di Bruno’s formula, an identity generalizing the chain rule, plays a significant role.

The author thanks her advisor, Thomas Kriete, for sharing his vision and for his continuous guidance, and Paul Bourdon for his insightful suggestions. She also thanks Vladimir Bolotnikov for sharing his ideas about boundary interpolation on the unit disk.

2. PRELIMINARIES

2.1. Generalized Chain Rule - Faà di Bruno’s Formula.

Faà di Bruno’s formula is an identity generalizing the chain rule that has been known since 1800. The following is the statement of the formula in combinatorial form.

Theorem 2.1 (Faà di Bruno’s formula). [19] *If g is analytic at z and f is analytic at $g(z)$, then*

$$(2.1) \quad (f \circ g)^{(k)}(z) = \sum_{\pi \in \Pi} f^{(|\pi|)}(g(z)) \cdot \prod_{B \in \pi} g^{(|B|)}(z),$$

where Π is the set of partitions of $\{1, \dots, k\}$.

We define the n^{th} order data of a function h at a point z to be the vector

$$D_n(h, z) = (h(z), h'(z), \dots, h^{(n)}(z)).$$

Note that as a consequence of Faà di Bruno’s formula, $D_k(f \circ g, z)$ is determined by $D_k(g, z)$ and $D_k(f, g(z))$ when g is analytic at z and f is analytic at $g(z)$. Furthermore, in this case we can rewrite Eq. (2.1) as

$$(2.2) \quad (f \circ g)^{(k)}(z) = f^{(k)}(g(z))g'(z)^k + F(D_{k-1}(f, g(z)), D_{k-1}(g, z)) + f'(g(z))g^{(k)}(z),$$

where the first term originates from the partition $\pi = \{\{1\}, \{2\}, \dots, \{k\}\}$, the last term originates from the partition $\pi = \{\{1, 2, \dots, k\}\}$, and F is defined by

$$F((a_0, a_1, \dots, a_{k-1}), (b_0, b_1, \dots, b_{k-1})) = \sum_{\substack{\pi \in \Pi \\ 1 < |\pi| < k}} a_{|\pi|} \cdot \prod_{B \in \pi} b_{|B|}.$$

2.2. Order of Contact.

Our treatment of order of contact is similar to that in [21]. We define the notion of order of contact with the boundary both in the context of the unit disk \mathbb{D} and in the context of the upper half-plane \mathbb{H} .

Definition 2.2. Let V be a neighborhood of some $\zeta \in \partial\mathbb{D}$ and let φ be analytic on $V \cap \mathbb{D}$ satisfying $\varphi(V \cap \mathbb{D}) \subset \mathbb{D}$. We say that φ has contact with $\partial\mathbb{D}$ at ζ of order $c > 0$ if the following conditions hold.

- (i) $\varphi(\zeta) := \lim_{z \rightarrow \zeta, z \in V \cap \mathbb{D}} \varphi(z)$ exists and $\varphi(\zeta) \in \partial\mathbb{D}$.
- (ii) $\frac{1 - |\varphi(e^{i\theta})|^2}{|\varphi(\zeta) - \varphi(e^{i\theta})|^c}$ is essentially bounded above and away from zero as $e^{i\theta} \rightarrow \zeta$.

A conformal mapping argument together with an application of Fatou's theorem shows that the non-tangential boundary values $\varphi(e^{i\theta})$ exist a.e. on $V \cap \partial\mathbb{D}$, and similarly for the half-plane definition below.

Definition 2.3. Let W be a neighborhood of 0 and let f be analytic on $W \cap \mathbb{H}$ satisfying $f(W \cap \mathbb{H}) \subset \mathbb{H}$. We say that f has contact with \mathbb{R} at 0 if the following conditions hold.

- (i) $f(0) := \lim_{z \rightarrow 0, z \in W \cap \mathbb{H}} f(z)$ exists and $f(0) \in \mathbb{R}$.
- (ii) $\frac{\operatorname{Im} f(x)}{|f(0) - f(x)|^c}$ is essentially bounded above and away from zero as $x \rightarrow 0$ in \mathbb{R} .

We transfer \mathbb{D} to \mathbb{H} using the family of conformal maps $\tau_\alpha: \mathbb{D} \rightarrow \mathbb{H}$ for $\alpha \in \partial\mathbb{D}$, defined by $\tau_\alpha: z \mapsto i \frac{\alpha - z}{\alpha + z}$. Note that φ has order of contact c with $\partial\mathbb{D}$ at ζ if and only if $f = \tau_{\varphi(\zeta)} \circ \varphi \circ \tau_\zeta^{-1}$ has order of contact c with \mathbb{R} at 0.

Order of contact is more easily understood in the context of the upper half-plane. To gain some intuition, suppose that f maps $W \cap \mathbb{H}$ into \mathbb{H} for some neighborhood W of 0 and is analytic at 0 with Taylor expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$ there. By taking imaginary parts, we see that f maps an interval in \mathbb{R} containing 0 into \mathbb{R} if and only if all the coefficients a_k are real. Otherwise, for $n = \min\{k: \operatorname{Im} a_k \neq 0\}$ we have that $0 < \operatorname{Im} f(x) \sim \operatorname{Im}(a_n)x^n$ as $x \rightarrow 0$ in \mathbb{R} . We see that n must be even, $\operatorname{Im} a_n > 0$, and

$$\frac{\operatorname{Im} f(x)}{|f(0) - f(x)|^n} \sim \frac{\operatorname{Im}(a_n)x^n}{|a_1|^n x^n} = \text{const},$$

so that f has order of contact n with \mathbb{R} at 0 (see discussion in [21]).

We use Faà di Bruno's Formula to determine the order of contact of composite maps which extend analytically to a neighborhood of the point of contact.

Proposition 2.4. *Let f_1 and f_2 be functions mapping $W \cap \mathbb{H}$ into \mathbb{H} for some neighborhood W of 0 that are analytic at 0, and suppose that f_1 fixes 0. Then the following statements hold.*

- (i) *If for $i = 1, 2$, f_i has order of contact N_i with \mathbb{R} at 0, then $f_2 \circ f_1$ has order of contact equal to $\min(N_1, N_2)$ with \mathbb{R} at 0.*
- (ii) *If one of f_1 and f_2 has order of contact N with \mathbb{R} at 0 and the other maps an interval of \mathbb{R} containing 0 into \mathbb{R} , then $f_2 \circ f_1$ has order of contact N with \mathbb{R} at 0.*

Proof. Let $f_1(z) = \sum_{j=0}^{\infty} a_j z^j$ and $f_2(z) = \sum_{j=0}^{\infty} b_j z^j$ be the Taylor series of f_1 and f_2 about 0. Let $N_1 = \min\{k: \operatorname{Im} a_k \neq 0\}$ and $N_2 = \min\{k: \operatorname{Im} b_k \neq 0\}$, allowing for ∞ in the case where all coefficients are real. Let $K = \min\{N_1, N_2\}$, so that K is finite under the assumption of either of the statements we aim to prove.

Since $a_1, \dots, a_{K-1}, b_1, \dots, b_{K-1} \in \mathbb{R}$, using Theorem 2.1 (Faà di Bruno's formula) and induction, we get that $(f_2 \circ f_1)'(0), \dots, (f_2 \circ f_1)^{(K-1)}(0)$ are all real valued. By Eq. (2.2) at $z = 0$ we have

$$(f_2 \circ f_1)^{(K)}(0) = f_2^{(K)}(0)f_1'(0)^K + F(D_{K-1}(f_2, 0), D_{K-1}(f_1, 0)) + f_2'(0)f_1^{(K)}(0),$$

and taking imaginary parts we get

$$\operatorname{Im}(f_2 \circ f_1)^{(K)}(0) = f_1'(0)^K \operatorname{Im} f_2^{(K)}(0) + f_2'(0) \operatorname{Im} f_1^{(K)}(0).$$

Since $a_1, b_1 > 0$, and $\operatorname{Im} a_K, \operatorname{Im} b_K \geq 0$ with at least one of them positive by the definition of K , it follows that $\operatorname{Im}(f_2 \circ f_1)^{(K)}(0) > 0$. Thus $f_2 \circ f_1$ has order of contact K with \mathbb{R} at 0. \square

2.3. The Class of Functions \mathcal{S} .

We work in the class of functions \mathcal{S} , introduced by Kriete-Moorhouse in [21], consisting of analytic self-maps φ of \mathbb{D} with certain properties of boundary regularity. The motivating model for a function in \mathcal{S} is an analytic self-map of \mathbb{D} which extends analytically to a neighborhood of $\overline{\mathbb{D}}$ and is not a finite Blaschke product. In particular, we restrict the number of points of significant contact with the unit circle, and require relatively nice behavior at these points of significant contact. To make a precise definition, we first discuss Clark measures. For $\zeta \in \partial\mathbb{D}$, we let $\varphi(\zeta)$ denote the non-tangential limit of φ at ζ , and $\varphi'(\zeta)$ denote the angular derivative of φ at ζ . If $\varphi'(\zeta)$ does not exist we say that $|\varphi'(\zeta)| = \infty$. If $\varphi'(\zeta)$ does exist, then $\varphi'(\zeta) = \bar{\zeta}\varphi(\zeta)|\varphi'(\zeta)|$, and in particular $\varphi'(\zeta) > 0$ if φ fixes ζ [15].

Let φ be an analytic self-map of \mathbb{D} . If $|\alpha| = 1$, there exists a finite positive Borel measure μ_α on $\partial\mathbb{D}$ such that

$$\frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = \operatorname{Re} \left(\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} \right) = \int_{\partial\mathbb{D}} P_z(e^{it}) d\mu_\alpha(t)$$

for z in \mathbb{D} , where $P_z(e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2}$ is the Poisson kernel at z . The existence of μ_α follows since the left side the equation above is a positive harmonic function. The measures μ_α are called the Clark measures of φ (see [11], [25]).

The singular part of the measure, μ_α^s , is carried by $\varphi^{-1}(\{\alpha\})$, the set of those $\zeta \in \partial\mathbb{D}$ where $\varphi(\zeta)$ exists and equals α . The measure μ_α^s is the sum of the pure point measure $\mu_\alpha^{pp} = \sum_{\varphi(\zeta)=\alpha} \frac{1}{|\varphi'(\zeta)|} \delta_\zeta$, where δ_ζ is the unit point mass at ζ , and a continuous singular measure μ_α^{cs} , either of which can be zero. We write $E(\varphi) = \bigcup_{|\alpha|=1} \operatorname{spt}(\mu_\alpha^s)$, where $\operatorname{spt}(\mu)$ denotes the closed support of a measure μ , and note that for any φ , $F(\varphi) = \{\zeta : \varphi \text{ has finite angular derivative at } \zeta\}$ is a subset of $E(\varphi)$. Furthermore, if $E(\varphi)$ is finite then the continuous singular measures μ_α^{cs} all vanish, and we get that $E(\varphi) = F(\varphi)$. For any φ , C_φ is a compact operator on H^2 if and only if $E(\varphi)$ is the empty set, see Sarason [24], Shapiro-Sundberg [28], and Cima-Matheson [11].

Definition 2.5. We define the class \mathcal{S} to be the set of analytic self-maps φ of \mathbb{D} satisfying the following conditions.

- (i) $|\varphi(e^{i\theta})| < 1$ a.e. on $\partial\mathbb{D}$.
- (ii) $E(\varphi)$ is a finite set, so that $E(\varphi) = F(\varphi)$.
- (iii) For each point $\zeta \in F(\varphi)$, there exists an even positive integer n such that φ has order of contact n at ζ , and complex numbers a_0, a_1, \dots, a_n with

$$\varphi(z) = a_0 + a_1(z - \zeta) + \dots + a_n(z - \zeta)^n + o(|z - \zeta|^n)$$

as $z \rightarrow \zeta$ unrestrictedly in \mathbb{D} .

Note: it can be shown that for any φ , (ii) implies (i).

For $\varphi \in \mathcal{S}$ with order of contact n at ζ , we define the derivatives of φ at ζ by

$$\varphi^{(j)}(\zeta) := \angle \lim_{z \rightarrow \zeta} \varphi^{(j)}(z) = j!a_j$$

for $j = 1, \dots, n$ and note that these non-tangential limits do exist (see the argument in [25, p 47]).

Proposition 2.6. *\mathcal{S} contains all self-maps of \mathbb{D} that extend analytically to a neighborhood of $\overline{\mathbb{D}}$ and are not finite Blaschke products.*

Proof. Let φ be such a map defined on a neighborhood V of $\overline{\mathbb{D}}$, and define $A = \{\zeta \in \partial\mathbb{D} : |\varphi(\zeta)| = 1\}$ to be the set of points where φ has contact with $\partial\mathbb{D}$. Then A consists of zeros of the analytic function $f(z) = \varphi(z) - \varphi_e(z)$, where $\varphi_e = \rho \circ \varphi \circ \rho$, and $\rho: z \mapsto 1/\bar{z}$ is inversion in the unit circle. To obtain a contradiction, suppose that A is infinite. Then f must be identically 0 on V and φ must be a finite Blaschke product, contradicting our assumption. Thus A is finite and $E(\varphi) \subset A$ is finite as well. Now let $\zeta \in F(\varphi) = E(\varphi)$. Note that φ is analytic at ζ and maps a small arc containing ζ onto a curve with contact with $\partial\mathbb{D}$ at exactly one point. Thus φ has finite (necessarily even) order of contact, say n , with $\partial\mathbb{D}$ at ζ . To complete the proof, we use the Taylor coefficients of φ at ζ to write

$$\varphi(z) = a_0 + a_1(z - \zeta) + \dots + a_n(z - \zeta)^n + o(|z - \zeta|^n).$$

□

2.4. The Denjoy-Wolff Point.

If φ is an analytic self-map of \mathbb{D} , not the identity and not an elliptic automorphism, then φ has a unique attractive fixed point ω in $\overline{\mathbb{D}}$. If ω lies on $\partial\mathbb{D}$, it is characterized by $\varphi(\omega) = \omega$ and $0 < \varphi'(\omega) \leq 1$. As above, $\varphi(\omega)$ is interpreted in the sense of nontangential limit and $\varphi'(\omega)$ is the angular derivative at ω ; see [15, Section 2.3].

3. THE BOUNDARY CARATHÉODORY-FEJÉR PROBLEM

The Carathéodory-Fejér problem [9, 4] is to determine whether a given finite sequence of complex numbers comprises the initial Taylor coefficients of an analytic map f mapping the unit disk \mathbb{D} to the upper half-plane \mathbb{H} . In this section, we explore a special case of a boundary version of the Carathéodory-Fejér problem studied by Agler-Lykova-Young in [1, 2], where the functions considered are analytic self-maps of \mathbb{H} . We note that Bolotnikov studies an alternative version of boundary interpolation, where the functions considered are self-maps of \mathbb{D} [5, 6].

For any x in \mathbb{R} we let \mathcal{P}_x denote the set of maps in \mathcal{P} that extend analytically to a neighborhood of x , where \mathcal{P} is the Pick class consisting of maps f analytic on \mathbb{H} which satisfy $\text{Im } f(z) \geq 0$ on \mathbb{H} . In [1], Agler-Lykova-Young study a boundary interpolation problem, denoted ∂CFP , where the interpolation node x lies on \mathbb{R} and solutions lie in \mathcal{P}_x . In the subsequent paper, [2], weaker solutions to ∂CFP , having non-tangential pseudo-Taylor expansions, are considered. For our purposes, unrestricted pseudo-Taylor expansions will suffice. We restrict attention to the case where solutions to ∂CFP have even order of contact n with \mathbb{R} at 0. The following is a consequence of [1, Theorem 1.2(2)] and [2, Theorem 5.2].

Theorem 3.1. [1, 2] *Let $n = 2m$ be an even positive integer, $a_0, \dots, a_{n-1} \in \mathbb{R}$ and $a_n \in \mathbb{H}$, and let $H_m(a_1, \dots, a_{n-1})$ be the Hankel matrix defined by*

$$H_m(a_1, \dots, a_{n-1}) = \begin{bmatrix} a_1 & a_2 & \dots & a_m \\ a_2 & a_3 & \dots & a_{m+1} \\ \cdot & \cdot & \dots & \cdot \\ a_m & a_{m+1} & \dots & a_{n-1} \end{bmatrix}.$$

Then the following are equivalent:

- (i) *There exists a function $f \in \mathcal{P}_0$ that has initial Taylor coefficients a_0, \dots, a_n at 0.*
- (ii) *There exists a function $f \in \mathcal{P}$ satisfying*

$$f(z) = a_0 + a_1 z + \dots + a_n z^n + o(|z|^n)$$

as $z \rightarrow 0$ unrestrictedly in \mathbb{H} .

- (iii) *$H_m(a_1, \dots, a_{n-1}) > 0$, i.e., this matrix is positive definite.*

3.1. Parametrization of Solutions to the Contact- n Case.

In [1], Agler-Lykova-Young give a parametrization of all solutions in the case where a_0, \dots, a_n are real. We apply the same techniques to the order of contact n case, that is, the case where $a_0, \dots, a_{n-1} \in \mathbb{R}$ and $a_n \in \mathbb{H}$, and arrive at a similar parametrization. The main tool used is a technique for passing from a function in the Pick class to a simpler one and back again due to G. Julia [20]. Reduction and augmentation (at 0) of a function are defined as follows.

Definition 3.2. For any non-constant function $f \in \mathcal{P}_0$ such that $f(0) \in \mathbb{R}$, we define the **reduction** of f (at 0) to be the function g on \mathbb{H} given by the equation

$$g(z) = -\frac{1}{f(z) - f(0)} + \frac{1}{f'(0)z}.$$

Definition 3.3. For any function $g \in \mathcal{P}_0$ and any $a_0 \in \mathbb{R}$, $a_1 > 0$, we define the **augmentation** of g (at 0) by a_0, a_1 to be the function f on \mathbb{H} given by

$$f(z) = a_0 + \frac{1}{\frac{1}{a_1 z} - g(z)}.$$

Reduction and augmentation preserve the Pick class (see [3, Theorem 3.4]) and are inverse operations for functions in \mathcal{P}_0 , that is,

- (i) if $f \in \mathcal{P}_0$ is non-constant and $f(0) \in \mathbb{R}$ then the reduction g of f is in \mathcal{P}_0 as well, and f is the augmentation of g by $f(0), f'(0)$;
- (ii) if $g \in \mathcal{P}_0$ and $a_0 \in \mathbb{R}$, $a_1 > 0$ then the augmentation f of g by a_0, a_1 is in \mathcal{P}_0 as well and satisfies $f(0) = a_0$ and $f'(0) = a_1$, and g is the reduction of f .

The relationship between the Taylor coefficients of a function and those of its reduction is explicitly expressed in [1, Proposition 2.5]. The following corollary to [1, Proposition 2.5] includes a statement contained in [1, Corollary 3.3].

Corollary 3.4. *Let $f \in \mathcal{P}_0$ satisfy $f'(0) > 0$, and let g be the reduction of f . Let the Taylor expansions of f and g about 0 be $f(z) = \sum_{j=0}^{\infty} a_j z^j$ and $g(z) = \sum_{j=0}^{\infty} b_j z^j$. Then the following statements hold.*

- (i) *For any $n \geq 2$, a_1, \dots, a_n determine b_0, \dots, b_{n-2} and in the other direction a_1, b_0, \dots, b_{n-2} determine a_2, \dots, a_n .*

- (ii) If $a_0 \in \mathbb{R}$ then for any $k \geq 2$, a_k is the first non-real Taylor coefficient of f if and only if b_{k-2} is the first non-real Taylor coefficient of g , that is to say, reduction reduces the order of contact by 2.
- (iii) $H_m(a_1, \dots, a_{2m-1}) > 0$ if and only if $H_{m-1}(b_1, \dots, b_{2m-3}) > 0$.

Proof. (i) and (ii) follow from [1, Proposition 2.5] and a calculation, and (iii) is contained in [1, Corollary 3.3]. \square

For a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we denote the corresponding linear fractional transformation by $L[A]$:

$$L[A]h = \frac{a_{11}h + a_{12}}{a_{21}h + a_{22}}.$$

With this notation, introduced in [1], we can express the augmentation f of g by a_0, a_1 by $f(z) = L[A(a_0, a_1)(z)]g(z)$, where $A(a_0, a_1)(z)$ is defined by

$$A(a_0, a_1)(z) = \begin{bmatrix} a_0 a_1 z & -a_0 - a_1 z \\ a_1 z & -1 \end{bmatrix}.$$

Note that composition of linear fractional transformations corresponds to matrix multiplication, and so this notation enables conversion of multiple augmentations into matrix multiplication.

Theorem 3.5. *Let $n = 2m$ be an even positive integer and let $a_0, \dots, a_{n-1} \in \mathbb{R}$ and $a_n \in \mathbb{H}$ be such that $H_m(a_1, \dots, a_{n-1}) > 0$. Let*

$$a_0 = a_0^{(0)}, a_0^{(1)}, \dots, a_0^{(m-1)} \in \mathbb{R}, \quad a_0^{(m)} \in \mathbb{H}, \quad a_1 = a_1^{(0)}, a_1^{(1)}, \dots, a_1^{(m-1)} > 0,$$

be the parameters determined by a_0, \dots, a_n via the procedure in the proof below. Then a function $f \in \mathcal{P}_0$ has initial Taylor coefficients a_0, \dots, a_n if and only if f is of the form

$$f(z) = L[A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z)]g(z)$$

where $g \in \mathcal{P}_0$ and satisfies $g(0) = a_0^{(m)}$.

Proof. Note that by Theorem 3.1, there exists a function $F_0 \in \mathcal{P}_0$ with initial Taylor coefficients a_0, \dots, a_n . We inductively define $F_{k+1} \in \mathcal{P}_0$ to be the reduction of F_k for $k = 1, \dots, m$, and $a_0^{(k)}, a_1^{(k)}, \dots$ to be the Taylor coefficients of F_k at 0. Notice that for each k , F_k is the augmentation of F_{k+1} by $a_0^{(k)}, a_1^{(k)}$, so that $F_k(z) = L[A(a_0^{(k)}, a_1^{(k)})]F_{k+1}(z)$, and so F_0 can be written as

$$F_0(z) = L[A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z)]F_m(z).$$

By Corollary 3.4(i), all the Taylor coefficients listed below are determined by a_0, \dots, a_n and do not depend on our choice of F_0 .

$$\begin{array}{ccccccc}
F_0 & \xrightarrow{\text{red}} & F_1 & \xrightarrow{\text{red}} & \dots & \xrightarrow{\text{red}} & F_{m-1} & \xrightarrow{\text{red}} & F_m \\
a_0 & & a_0^{(1)} & & \dots & & a_0^{(m-1)} & & \mathbf{a}_0^{(\mathbf{m})} \\
a_1 & & a_1^{(1)} & & \dots & & a_1^{(m-1)} & & \\
a_2 & & a_2^{(1)} & & \dots & & \mathbf{a}_2^{(\mathbf{m}-1)} & & \\
\vdots & & \vdots & & \ddots & & & & \\
a_{n-2} & & \mathbf{a}_{n-2}^{(1)} & & & & & & \\
a_{n-1} & & & & & & & & \\
\mathbf{a}_n & & & & & & & &
\end{array}$$

In general, for $k = 1, \dots, m$ we have that $a_0^{(k)}, \dots, a_{n-2k}^{(k)}$ are determined. It follows from Corollary 3.4(ii) that $a_0^{(k)}, \dots, a_{n-2k-1}^{(k)} \in \mathbb{R}$ and $a_{n-2k}^{(k)} \in \mathbb{H}$, i.e., all the non-bold coefficients above are in \mathbb{R} and all the bold coefficient are in \mathbb{H} . In particular, we get that

$$a_0 = a_0^{(0)}, a_0^{(1)}, \dots, a_0^{(m-1)} \in \mathbb{R}, \quad a_0^{(m)} \in \mathbb{H}.$$

Additionally, Corollary 3.4(iii) implies that for $k = 1, \dots, m-1$ the Hankel matrix $H_{m-k}(a_1^{(k)}, \dots, a_{n-2k-1}^{(k)})$ is positive, and so in particular we have

$$a_1 = a_1^{(0)}, a_1^{(1)}, \dots, a_1^{(m-1)} > 0.$$

Note that since F_0, \dots, F_{m-1} are all real valued at 0 and non-constant, taking the above reductions makes sense.

To prove the forward implication, suppose $f = f_0 \in \mathcal{P}_0$ has initial Taylor coefficients a_0, \dots, a_n , and let f_k denote the k^{th} reduction of f_0 . As discussed above, the first $n - 2k + 1$ Taylor coefficients of f_k are $a_0^{(k)}, \dots, a_{n-2k}^{(k)}$, and so f_0 can be written as

$$f_0(z) = L[A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z)]f_m(z),$$

with f_m satisfying $f_m(0) = a_0^{(m)}$.

To prove the backward implication, let $g = f_m \in \mathcal{P}_0$ satisfy $g(0) = a_0^{(m)}$ and inductively define f_k for $k = m-1, \dots, 0$ to be the augmentation of f_{k+1} by $a_0^{(k)}, a_1^{(k)}$, i.e., $f_k(z) = L[A(a_0^{(k)}, a_1^{(k)})]f_{k+1}(z)$. In order to complete the proof note that it follows from Corollary 3.4(i) that for each k the first $n - 2k + 1$ Taylor coefficients of f_k are again $a_0^{(k)}, \dots, a_{n-2k}^{(k)}$. \square

Note that by multiplying the matrices in Theorem 3.5, we get that functions $f \in \mathcal{P}_0$ with initial Taylor coefficients a_0, \dots, a_n are of the form

$$f(z) = \frac{p(z)h(z) + q(z)}{r(z)h(z) + s(z)},$$

where p, q, r, s are polynomials with real coefficients of degree at most m determined by a_0, \dots, a_n and $h \in \mathcal{P}_0$ satisfies $h(0) = h_0$, where h_0 is determined by a_0, \dots, a_n . Additionally, a calculation of determinants shows that for some $K > 0$ the polynomials p, q, r, s satisfy $(ps - qr)(z) = Kz^n$.

We let the Taylor coefficients a_0, \dots, a_n vary to obtain the following corollary.

Proposition 3.6. *Let $f \in \mathcal{P}_0$. Then f has order of contact $n = 2m$ with \mathbb{R} at 0 if and only if f is of the form*

$$f(z) = L[A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z)]g(z),$$

where $g \in \mathcal{P}_0$ satisfies $g(0) \in \mathbb{H}$ and

$$a_0^{(0)}, a_0^{(1)}, \dots, a_0^{(m-1)} \in \mathbb{R}, \quad a_1^{(0)}, a_1^{(1)}, \dots, a_1^{(m-1)} > 0.$$

Furthermore, for any $f \in \mathcal{P}_0$ this representation is unique.

Proof. For the first direction, assume f has order of contact $n = 2m$ with \mathbb{R} at 0, and let $a_0, \dots, a_{n-1} \in \mathbb{R}$ and $a_n \in \mathbb{H}$ be the initial Taylor coefficients of f at 0. Then by Theorem 3.1 we get that $H_m(a_1, \dots, a_{n-1}) > 0$, and so by Theorem 3.5 f is of the desired form.

For the other direction note that any f of this form is obtained by applying m augmentations to $g \in \mathcal{P}_0$, and so $f \in \mathcal{P}_0$. Recall that by Corollary 3.4(ii), augmentation increases the order of contact with \mathbb{R} by 2, and so $g(0) \in \mathbb{H}$ implies that f has order of contact $2m = n$.

To see uniqueness, notice that if

$$f(z) = L[A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z)]g(z)$$

then $a_0^{(k)} = f_k(0)$, $a_1^{(k)} = f'_k(0)$ and $g = f_m$ where f_k denotes the k^{th} reduction of f . \square

3.2. Rational Functions with Specified Taylor Coefficients.

We turn our attention to construction of simple solutions to ∂CFP . We construct rational maps $f \in \mathcal{P}_0$ which map 0 into \mathbb{R} and the rest of $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ into \mathbb{H} and have specified initial Taylor coefficients at 0. Additionally we require that 0 be a regular value for f , i.e., that $f^{-1}(\{0\})$ consists of d distinct points where d is the degree of f .

Observation 3.7. Reduction and augmentation preserve rationality and boundary behavior of functions in the sense that if g is the reduction of f , then the following assertions hold.

- (i) g is a rational function of degree d if and only if f a rational function of degree $d + 1$.
- (ii) For any $x \in \mathbb{R} \setminus \{0\}$, $\text{Im } g(x) > 0$ if and only if $\text{Im } f(x) > 0$.

Proposition 3.8. *Let f be in \mathcal{P} and suppose that for some $n = 2m$, f has pseudo Taylor coefficients*

$$\lim_{z \rightarrow 0} \frac{f^{(k)}(z)}{k!} = a_k \quad \text{for } k = 0, 1, \dots, n,$$

where the limits are taken unrestrictedly in \mathbb{H} , such that $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$ and $a_n \in \mathbb{H}$. Then there exists a degree $m + 1$ rational function $f_0 \in \mathcal{P}_0$ that has 0 as a regular value, maps $\widehat{\mathbb{R}} \setminus \{0\}$ into \mathbb{H} and has initial Taylor coefficients a_0, \dots, a_n at $z = 0$.

Proof. By Theorem 3.1, $H_m(a_1, \dots, a_{n-1}) > 0$ and so by Theorem 3.5 we have that for any $g \in \mathcal{P}_0$ with $g(0) = a_0^{(m)}$,

$$F(z) = L[A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z)]g(z),$$

is in \mathcal{P}_0 with the desired Taylor coefficients. Here $a_0^{(0)} = a_0, a_0^{(1)}, \dots, a_0^{(m-1)} \in \mathbb{R}$, $a_0^{(m)} \in \mathbb{H}$ and $a_1^{(0)} = a_1, a_1^{(1)}, \dots, a_1^{(m-1)} > 0$ are determined by a_0, \dots, a_n .

For any $w \in \mathbb{H}$, we define $g_w(z) = L[A(a_0^{(m)}, 1)(z)]w = a_0^{(m)} + \frac{z}{1-wz}$, so that g is a degree 1 rational function in \mathcal{P}_0 . Note that g_w can be written as the sum of $\text{Im } a_0^{(m)}$ and an augmentation of the constant function w , and so by the observation above g_w maps $\widehat{\mathbb{R}}$ into \mathbb{H} . Let F_w denote the function F above resulting from the choice $g = g_w$ and note that F_w is obtained by applying m augmentations to g_w . Thus F_w is a degree $m+1$ rational function mapping $\mathbb{R} \setminus \{0\}$ into \mathbb{H} .

It remains to find a $w \in \mathbb{H}$ such that 0 is a regular value for F_w . We write

$$\begin{bmatrix} p(z) & q(z) \\ r(z) & s(z) \end{bmatrix} = A(a_0^{(0)}, a_1^{(0)})(z) \cdots A(a_0^{(m-1)}, a_1^{(m-1)})(z) A(a_0^{(m)}, 1)(z),$$

so that p, q, r, s are polynomials of degree at most $m+1$, and $F_w(z) = \frac{p(z)w + q(z)}{r(z)w + s(z)}$. Note that a calculation of determinants shows that p, q, r, s satisfy $(ps - qr)(z) = (a_1^{(0)})^2 z^2 \cdots (a_1^{(m)})^2 z^2 = Kz^{n+2}$ for some $K \neq 0$, and recall that F_w has degree $m+1$. An elementary argument considering degrees and common factors of polynomials shows that $F_w(z) = 0$ has $m+1$ distinct solutions for all but finitely many choices of $w \in \mathbb{C}$. \square

Note that if we forgo the requirement that 0 be a regular value of f_0 , the choice $g(z) \equiv a_0^{(m)}$ in the above proof suffices, and the degree of f_0 is reduced to m .

4. RELATIONS IN THE CALKIN ALGEBRA

In [21], Kriete-Moorhouse investigate compactness of linear combinations of composition operators where the inducing maps lie in the class \mathcal{S} . We review some definitions and results from [21], then apply our results from Section 3 to obtain a decomposition of such a composition operator, modulo the ideal \mathcal{K} of compact operators, into a sum of composition operators induced by basic or rational functions (see Definition 4.2).

Additionally, we review a result from [21] regarding weighted composition operators modulo \mathcal{K} and use this result in the proof of a similar result concerning weighted adjoints of composition operators.

4.1. Linear Relations in the Calkin Algebra for Composition Operators.

In [21], Kriete-Moorhouse show that information relating to compactness of a linear combination of compositions operators $c_1 C_{\varphi_1} + \dots + c_r C_{\varphi_r}$, where $\varphi_1, \dots, \varphi_r \in \mathcal{S}$, is carried by the behavior of the functions φ_j at their points of contact with the unit circle. More precisely, the relevant information for φ at a point of contact ζ is $D_n(\varphi, \zeta) = (\varphi(\zeta), \varphi'(\zeta), \dots, \varphi^{(n)}(\zeta))$, where n is the order of contact of φ with the unit circle at ζ . The following result determines compactness of a linear combination of composition operators for operators induced by functions in \mathcal{S} .

Theorem 4.1. [21, Theorem 5.13] *Let $\varphi_1, \dots, \varphi_r$ in \mathcal{S} and write F for the union $F(\varphi_1) \cup \dots \cup F(\varphi_r)$, a finite set. For ζ in F and $k = 2, 4, 6, \dots$, let*

$$\mathbb{N}_k(\zeta) = \{j : F(\varphi_j) \text{ contains } \zeta \text{ and } k \text{ is the order of contact of } \varphi_j \text{ at } \zeta\}$$

and let

$$\mathcal{E}_k(\zeta) = \{D_k(\varphi_j, \zeta) : j \text{ is in } \mathbb{N}_k(\zeta)\}.$$

Given complex numbers c_1, \dots, c_r , the following are equivalent:

- (i) $c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_r}$ is compact;
- (ii) $\sum_{\substack{j \in \mathbb{N}_k(\zeta) \\ D_k(\varphi_j, \zeta) = d}} c_j = 0$ for every ζ in F , every even $k \geq 2$ and every d in $\mathcal{E}_k(\zeta)$.

Our goal is to decompose C_φ , modulo \mathcal{K} , into a sum of composition operators induced by basic functions.

Definition 4.2. A function φ analytic on \mathbb{D} is a basic function with contact at ζ if the following hold.

- (i) φ is a rational function mapping the unit disk \mathbb{D} into itself.
- (ii) $\varphi(\zeta)$ is on the unit circle, and φ maps the rest of the unit circle into \mathbb{D} .
- (iii) $\varphi(\zeta)$ is a regular value for φ .

Lemma 4.3. Let $\varphi \in \mathcal{S}$ and $\zeta \in F(\varphi)$. Let $n = 2m$ denote the order of contact of φ with $\partial\mathbb{D}$ at ζ . Then there exists a degree $m + 1$ basic function φ_0 with order of contact n with $\partial\mathbb{D}$ at ζ which satisfies $D_n(\varphi_0, \zeta) = D_n(\varphi, \zeta)$.

Proof. We begin by defining $\lambda = \varphi(\zeta)$ and $f = \tau_\lambda \circ \varphi \circ \tau_\zeta^{-1}$. Then $f \in \mathcal{P}$ and since $\varphi \in \mathcal{S}$ has contact with $\partial\mathbb{D}$ of order n at ζ , there exist $a_1, a_2, \dots, a_{n-1} \in \mathbb{R}$ and $a_n \in \mathbb{H}$ such that f satisfies

$$f(z) = 0 + a_1(z - x) + \dots + a_n(z - x)^n + o(|z - x|^n)$$

as $z \rightarrow 0$ unrestrictedly in \mathbb{H} . By Proposition 3.8, there exists a degree $m + 1$ rational function $F \in \mathcal{P}_0$ that maps $\widehat{\mathbb{R}} \setminus \{0\}$ into \mathbb{H} , has initial Taylor coefficients $a_0 = 0, a_1, \dots, a_n$ at $z = 0$, and has $z = 0$ as a regular value.

We define $\varphi_0 = \tau_\lambda^{-1} \circ F \circ \tau_\zeta$ and get that φ_0 is a degree $m + 1$ basic function with order contact n with $\partial\mathbb{D}$ at ζ . Note that as a consequence of Theorem 2.1 (Faà di Bruno's formula), $D_n(F, 0) = D_n(f, 0)$ implies that φ_0 satisfies $D_n(\varphi_0, \zeta) = D_n(\varphi, \zeta)$ as desired. \square

Theorem 4.4. Let $\varphi \in \mathcal{S}$ with $F(\varphi) = \{\zeta_1, \dots, \zeta_r\}$ and let $n_j = 2m_j$ denote the order of contact of φ with the unit circle at ζ_j . Then there exists a decomposition,

$$C_\varphi \equiv C_{\varphi_1} + \dots + C_{\varphi_r} \pmod{\mathcal{K}},$$

where for each $j = 1, \dots, r$, φ_j is a basic function of degree $m_j + 1$ which has contact of order n_j at ζ_j and satisfies $D_{n_j}(\varphi_j, \zeta_j) = D_{n_j}(\varphi, \zeta_j)$.

Proof. Existence of $\varphi_1, \dots, \varphi_r$ follows from Lemma 4.3. The result follows by applying Theorem 4.1 to $\varphi, \varphi_1, \dots, \varphi_r$ with constants $1, -1, \dots, -1$. \square

Note that we can reduce the degree of the rational maps $\varphi_1, \dots, \varphi_r$ in the above decomposition to m_1, \dots, m_r respectively by relaxing the condition that these functions be basic and requiring only that they be rational self-maps of \mathbb{D} having contact with $\partial\mathbb{D}$ at exactly one point.

4.2. Weighted Composition Operators and Adjoints in the Calkin Algebra.

Given a bounded measurable function w on $\partial\mathbb{D}$, we consider the multiplication operator $M_w: f \mapsto wf$ which can be viewed as mapping L^2 to L^2 , H^2 to L^2 , or if w is in H^∞ , H^2 to H^2 . Additionally, we consider the Toeplitz operator $T_w = PM_w|_{H^2}$ (where P is the orthogonal projection of L^2 onto H^2) which maps H^2 to H^2 regardless of the choice of w .

In [21], it is shown that the coset of the weighted composition operator $M_w C_\varphi$ modulo the subspace of compact operators from H^2 to L^2 (also denoted by \mathcal{K} here for convenience) is in some sense determined by the values of w on $E(\varphi)$. For the special case that $\varphi \in \mathcal{S}$ has contact with $\partial\mathbb{D}$ at exactly one point, we have the following corollary.

Corollary 4.5. *Let $\varphi \in \mathcal{S}$ be such that $F(\varphi) = \{\zeta\}$. Suppose w is a bounded measurable function on $\partial\mathbb{D}$ such that w is continuous at ζ . Then the H^2 to L^2 operator $M_w C_\varphi$ satisfies*

$$M_w C_\varphi \equiv w(\zeta) C_\varphi \pmod{\mathcal{K}}.$$

Proof. Let $v(z) := w(z) - w(\zeta)$. Then v is bounded on $\partial\mathbb{D}$, continuous at ζ and satisfies $v \equiv 0$ on $E(\varphi) = F(\varphi) = \{\zeta\}$. Thus by [21, Theorem 3.1], $M_v C_\varphi = M_w C_\varphi - w(\zeta) C_\varphi$ is compact. \square

In Proposition 4.7 we prove a similar result for weighted adjoints of composition operators. The proof relies on existence of an H^∞ function which satisfies several boundary conditions.

Lemma 4.6. *Let I be an open arc in $\partial\mathbb{D}$ and suppose $\lambda \in I$. Let v be a non-negative bounded function on $\partial\mathbb{D}$ which is continuous on I , continuously differentiable on $I \setminus \{\lambda\}$ and satisfies $v(\lambda) = 0$. Then there exists an analytic function b on \mathbb{D} which extends continuously to $\overline{\mathbb{D}}$ and satisfies both $b(\lambda) = 0$ and $|b(e^{i\theta})| \geq v(e^{i\theta})$ on $\partial\mathbb{D}$.*

Proof. Let J be a closed sub-interval of I whose interior contains λ , and define u on J by $u(e^{i\theta}) = v(e^{i\theta}) + |e^{i\theta} - \lambda|$ for $e^{i\theta} \in J$. We extend u to all of $\partial\mathbb{D}$ in such a way that u is continuously differentiable on $\partial\mathbb{D} \setminus \{\lambda\}$ and satisfies

$$u(e^{i\theta}) \geq v(e^{i\theta}) + |e^{i\theta} - \lambda|$$

for all $e^{i\theta} \in \partial\mathbb{D}$. Then $\log u$ is integrable on $\partial\mathbb{D}$ and so we can define an analytic function h on \mathbb{D} by the Herglotz integral

$$h(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log u(e^{i\theta}) \frac{d\theta}{2\pi}.$$

Note that since $\log u$ is continuously differential on $\partial\mathbb{D} \setminus \{\lambda\}$, h extends continuously to $\partial\mathbb{D} \setminus \{\lambda\}$, see [18, pp 78-80].

Let $b = e^h$, so that b is a bounded analytic function on \mathbb{D} that extends continuously to $\partial\mathbb{D} \setminus \{\lambda\}$. Moreover,

$$|b(z)| = \exp \left(\int_0^{2\pi} P_z(e^{i\theta}) \log u(e^{i\theta}) \frac{d\theta}{2\pi} \right),$$

where $P_z(e^{i\theta})$ is the Poisson kernel at z . Since $\log u(e^{i\theta}) \rightarrow -\infty$ as $e^{i\theta} \rightarrow \lambda$, standard estimates on P_z show that $b(z) \rightarrow 0$ as $z \rightarrow \lambda$. Thus b extends continuously to all of $\partial\mathbb{D}$ with $b(\lambda) = 0$. Finally, we note that $|b(e^{i\theta})| = |u(e^{i\theta})| \geq v(e^{i\theta})$ on $\partial\mathbb{D}$. \square

Proposition 4.7. *Let $\varphi \in \mathcal{S}$ be such that $F(\varphi) = \{\zeta\}$ and denote $\lambda = \varphi(\zeta)$. Suppose that w is a bounded measurable function on $\partial\mathbb{D}$ such that w is continuous on I and continuously differentiable on $I \setminus \{\lambda\}$ for some open arc I in $\partial\mathbb{D}$ containing λ . Then*

$$M_w C_\varphi^* \equiv w(\lambda) C_\varphi^* \pmod{\mathcal{K}},$$

where M_w is viewed as an operator from H^2 to L^2 , and in particular,

$$T_w C_\varphi^* \equiv w(\lambda) C_\varphi^* \pmod{\mathcal{K}}.$$

Proof. It suffices to prove that $M_v C_\varphi^*$ is compact where $v = w - w(\lambda)$ since

$$M_w C_\varphi^* = (M_{w-w(\lambda)} + M_{w(\lambda)}) C_\varphi^* = M_v C_\varphi^* + w(\lambda) C_\varphi^*.$$

Note that v is continuous on I and continuously differentiable on $I \setminus \{\lambda\}$ and satisfies $v(\lambda) = 0$, and so by Lemma 4.6 there exists a $b \in H^\infty$ which extends continuously to $\partial\mathbb{D}$ and satisfies $b(\lambda) = 0$ and $|b(e^{i\theta})| \geq v(e^{i\theta})$ on $\partial\mathbb{D}$. We get that for all $f \in H^2$,

$$\begin{aligned} \|M_v C_\varphi^* f\|^2 &= \int_0^{2\pi} |v(e^{i\theta})|^2 |(C_\varphi^* f)(e^{i\theta})|^2 \frac{d\theta}{2\pi} \\ &\leq \int_0^{2\pi} |\overline{b(e^{i\theta})}|^2 |(C_\varphi^* f)(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \|M_{\bar{b}} C_\varphi^* f\|^2, \end{aligned}$$

and so it suffices to show that $M_{\bar{b}} C_\varphi^*$ is a compact operator from H^2 to L^2 . We write

$$M_{\bar{b}} C_\varphi^* = P M_{\bar{b}} C_\varphi^* + (I - P) M_{\bar{b}} C_\varphi^*$$

and show that both terms on the right hand side are compact.

We first note that since \bar{b} is a continuous function on $\partial\mathbb{D}$, the L^2 operator $(I - P) M_{\bar{b}} P$ is compact (see the version of Hartman's theorem in [23, p 214, Theorem 2.2.5]). Thus, the term $(I - P) M_{\bar{b}} C_\varphi^* = (I - P) M_{\bar{b}} P C_\varphi^*$ is compact from H^2 to L^2 .

We now show $P M_{\bar{b}} C_\varphi^* = T_{\bar{b}} C_\varphi^*$ is compact on H^2 by looking at its adjoint $(T_{\bar{b}} C_\varphi^*)^* = C_\varphi T_b$. Since $b \in H^\infty$, we have $C_\varphi T_b = C_\varphi M_b = M_{b \circ \varphi} C_\varphi$. We wish to apply Corollary 4.5 to $M_{b \circ \varphi} C_\varphi$. Recall that $\varphi \in \mathcal{S}$ with $F(\varphi) = \{\zeta\}$, and note that although the non-tangential boundary function $\varphi(e^{i\theta})$ is in general defined only almost everywhere, we can extend it to all of $\partial\mathbb{D}$ by setting $\varphi(e^{i\theta}) = \varphi(\zeta)$ on the remaining set of measure zero. It follows from the definition of \mathcal{S} that this extension (which we also call φ) is continuous at ζ . Thus $b \circ \varphi$ is continuous at ζ and $b(\varphi(\zeta)) = b(\lambda) = 0$, and so Corollary 4.5 implies that $M_{b \circ \varphi} C_\varphi$ is compact as desired. \square

5. ADJOINT FORMULA FOR RATIONALLY INDUCED COMPOSITION OPERATORS

Recent work of Cowen-Gallardo [14], Hammond-Moorhouse-Robbins [17] and Bourdon-Shapiro [8] has produced pointwise formulas for C_φ^* , where the inducing map φ is rational. The constituent parts of these pointwise formulas contain multiple-valued analytic functions which do not necessarily represent well-defined operators individually. We show how to work with these pointwise formulas to produce legitimate operator equations involving C_φ^* for the rational case. We then consider the case where φ is basic and reduce our equations to the Calkin algebra.

5.1. From Pointwise Formula To Operator Equation.

Let φ be a rational self-map of \mathbb{D} of degree d . We associate with φ its exterior map $\varphi_e := \rho \circ \varphi \circ \rho$, where $\rho: z \rightarrow 1/\bar{z}$ is the inversion in the unit circle. Then φ_e maps $\mathbb{D}_e := \{z \in \mathbb{C}: |z| > 1\}$ into itself, and so $\varphi_e^{-1}(\mathbb{D}) \subset \mathbb{D}$.

For any simply connected domain V consisting of regular values of φ_e , there exist d distinct branches $\sigma_1, \dots, \sigma_d$ of φ_e^{-1} defined on V , and we have that the sets $\sigma_1(V), \dots, \sigma_d(V)$ are pairwise disjoint (see [8]). Note that one possible choice of V

is the unit disk with radial slits from each critical value of φ_e to the unit circle removed. A choice that may be much smaller but sufficient for our needs is a small neighborhood of a regular value of φ_e .

We use the following variant of the pointwise formula for C_φ^* introduced by Bourdon-Shapiro in [8].

Proposition 5.1. [8, Corollary 8] *Suppose that V is a set on which d distinct branches $\sigma_1, \dots, \sigma_d$ of φ_e^{-1} are defined. Then for all $f \in H^2$ and all $z \in V \cap \mathbb{D}$,*

$$(5.1) \quad C_\varphi^* f(z) = \frac{f(0)}{1 - \overline{\varphi(0)}z} + \sum_{j=1}^d z \sigma_j'(z) S^* f(\sigma_j(z)),$$

where S^* is the adjoint of the shift operator S defined by $(Sf)(z) = zf(z)$.

Note that Eq. (5.1) can be rewritten, at least formally, as

$$(5.2) \quad C_\varphi^* = \Lambda + \sum_{j=1}^d M_{h_j} C_{\sigma_j} S^*,$$

where $h_j(z) = z \sigma_j'(z)$, and Λ is the rank one operator defined by $\Lambda(f) := \frac{f(0)}{1 - \overline{\varphi(0)}z}$. However, the maps σ_j are not in general analytic on all of \mathbb{D} , and so the operators on the right hand side are not in general “legitimate” operators.

Bourdon-Shapiro define φ as outer regular when its critical values all lie in \mathbb{D} . Note that for outer regular functions φ we can choose $V = r\mathbb{D}$ for some $r > 1$. Then, restricting domains to \mathbb{D} , we have that $\sigma_1, \dots, \sigma_d$ are analytic self-maps of \mathbb{D} and h_1, \dots, h_d are H^∞ functions. Thus, equation Eq. (5.2) is a legitimate operator equation in the outer regular case [8, Theorem 13(a)]. Unfortunately, the outer regular case is only possible for order of contact 2 functions.

Proposition 5.2. *If φ is a rational self-map of \mathbb{D} having order of contact $n > 2$ with the unit circle at ζ , then φ is not outer regular.*

Proof. Let φ be as in the assumption and in order to obtain a contradiction suppose that φ is outer regular. Then there exists a branch σ of φ_e^{-1} mapping $\varphi(\zeta)$ to ζ and defined on all of \mathbb{D} . We transfer the maps to the upper half-plane and work with $\tilde{\varphi} = \tau_{\varphi(\zeta)}^{-1} \circ \varphi \circ \tau_\zeta$, $\tilde{\varphi}_e = \tau_{\varphi(\zeta)}^{-1} \circ \varphi_e \circ \tau_\zeta$ and $\tilde{\sigma} = \tau_\zeta^{-1} \circ \sigma \circ \tau_{\varphi(\zeta)}$, noting that all three maps fix 0 and $\tilde{\varphi}, \tilde{\sigma} \in \mathcal{P}_0$.

Let a_0, a_1, a_2, a_3 denote the initial Taylor coefficients of $\tilde{\varphi}$ at $z = 0$, and b_0, b_1, b_2, b_3 denote those of $\tilde{\sigma}$. In Proposition 5.9, we show that σ has order of contact n at $\varphi(\zeta)$, and so by Theorem 3.1 the Hankel matrices $H_m(a_1, \dots, a_{n-1})$ and $H_m(b_1, \dots, b_{n-1})$ are both positive, and in particular, their 2nd leading principal minors, $a_1 a_3 - a_2^2$ and $b_1 b_3 - b_2^2$, are positive.

In Lemma 5.7 we show that $\tilde{\varphi}$ and $\tilde{\varphi}_e$ have equal Taylor coefficients a_0, \dots, a_{n-1} . Noting that $\tilde{\varphi}_e \circ \tilde{\sigma} = id$, we can therefore express b_1, b_2, b_3 in terms of a_1, a_2, a_3 using Theorem 2.1 (Faà di Bruno’s formula). We reach a contradiction by calculating that $b_1 b_3 - b_2^2 = \frac{-(a_1 a_3 - a_2^2)}{a_1^6} < 0$. \square

5.2. Generalized Adjoint Formula in the Calkin Algebra.

In general, the set V on which $\sigma_1, \dots, \sigma_d$ are analytic can not be chosen to contain all of \mathbb{D} , and so the formal operators $C_{\sigma_1}, \dots, C_{\sigma_d}$ in Eq. (5.2) are not legitimate operators. We overcome this difficulty by pre-composing with a map ψ with image

contained in $V \cap \mathbb{D}$ to obtain analytic self-maps of \mathbb{D} , $\sigma_1 \circ \psi, \dots, \sigma_d \circ \psi$. This enables us to write a legitimate operator formula for $C_\psi C_\varphi^*$.

Proposition 5.3. *Let ψ be an analytic self-map of \mathbb{D} satisfying $\overline{\psi(\mathbb{D})} \subset V$. Then*

$$C_\psi C_\varphi^* = C_\psi \Lambda + \sum_{j=1}^d M_{h_j \circ \psi} C_{\sigma_j \circ \psi} S^*$$

where $h_j(z) = z\sigma'_j(z)$, S^* is the adjoint of the shift operator and Λ is the rank one operator defined by $\Lambda(f) := \frac{f(0)}{1-\varphi(0)z}$.

In particular, $h_j \circ \psi$ are H^∞ functions and $\sigma_j \circ \psi$ are analytic self-maps of \mathbb{D} .

Proof. Since ψ maps \mathbb{D} into V and $\sigma_1, \dots, \sigma_d$ are analytic on V , we get that $\sigma_1 \circ \psi, \dots, \sigma_d \circ \psi$ are analytic self-maps of \mathbb{D} (recall that $\varphi_e^{-1}(\mathbb{D}) \subset \mathbb{D}$). The functions $\sigma'_1, \dots, \sigma'_d$ are analytic on V and so bounded on $\psi(\mathbb{D})$, so $h_1 \circ \psi, \dots, h_d \circ \psi$ are H^∞ functions. To complete the proof, note that by the pointwise formula given in Proposition 5.1, for all $f \in H^2$ and all $z \in \mathbb{D}$ we have

$$\begin{aligned} C_\psi C_\varphi^* f(z) &= C_\varphi^* f(\psi(z)) = \frac{f(0)}{1 - \overline{\varphi(0)}\psi(z)} + \sum_{j=1}^d \psi(z)\sigma'_j(\psi(z))S^* f(\sigma_j(\psi(z))) \\ &= (C_\psi \Lambda f)(z) + \sum_{j=1}^d (M_{h_j \circ \psi} C_{\sigma_j \circ \psi} S^* f)(z). \end{aligned}$$

□

We turn to the case where φ is a basic function of degree d with order of contact n at ζ . We denote $\lambda = \varphi(\zeta)$ and note that $\rho(\lambda) = \lambda$ is a regular value for φ_e , and so there exists a neighborhood $V(\varphi)$ of λ consisting of regular values of φ_e . Recall that $\sigma_1(V(\varphi)), \dots, \sigma_d(V(\varphi))$ are pairwise disjoint, and let σ denote the unique branch of φ_e^{-1} that maps λ to ζ . The following is a generalization of [8, Corollary 15].

Lemma 5.4. *Let φ be a basic function with contact with $\partial\mathbb{D}$ at ζ , and let ψ be an analytic self map of \mathbb{D} satisfying $\overline{\psi(\mathbb{D})} \subset V(\varphi)$. Then*

$$C_\psi C_\varphi^* \equiv M_{h \circ \psi} C_{\sigma \circ \psi} S^* \pmod{\mathcal{K}}$$

where $h(z) := z\sigma'(z)$.

In particular, $h \circ \psi$ is an $H^\infty(\mathbb{D})$ function and $\sigma \circ \psi$ is an analytic self-map of \mathbb{D} .

Proof. Notice that since ρ is the identity on $\partial\mathbb{D}$, φ_e maps exactly one point in $\partial\mathbb{D}$ (the point ζ) into $\partial\mathbb{D}$ and so $\varphi_e^{-1}(\partial\mathbb{D}) = \{\zeta\}$. We also have that the branches $\sigma_1, \dots, \sigma_d$ map $\overline{\psi(\mathbb{D})}$ to pairwise disjoint closed subsets of \mathbb{D} . Thus, for $\sigma_j \neq \sigma$ the closed set $\sigma_j(\overline{\psi(\mathbb{D})})$ does not intersect $\partial\mathbb{D}$ and so $\|\sigma_j \circ \psi\|_\infty < 1$ and we get that the composition operator $C_{\sigma_j \circ \psi}$ is compact.

Noting that $\Lambda: f \rightarrow \frac{f(0)}{1-\varphi(0)z}$ is rank one, reducing Proposition 5.3 modulo the compacts gives

$$C_\psi C_\varphi^* \equiv M_{h \circ \psi} C_{\sigma \circ \psi} S^* \pmod{\mathcal{K}}$$

where h is defined by $h(z) = z\sigma'(z)$, and by the same proposition h satisfies $h \circ \psi$ is an $H^\infty(\mathbb{D})$ function and $\sigma \circ \psi$ is an analytic self-map of \mathbb{D} . □

For a basic function φ , there exists a neighborhood $W(\varphi)$ of λ such that $\overline{W(\varphi)} \subset V(\varphi)$, and on which σ is bounded away from zero (recall that $\sigma(\lambda) = \zeta$). Restricting to this neighborhood enables the removal of S^* from the formula. In essence, working modulo \mathcal{K} , we transfer a term originating from a summand in the variant formula (Proposition 5.1), to a term resembling a summand in the original formula [17, Theorem 7].

Lemma 5.5. *Let φ be a basic function with contact with $\partial\mathbb{D}$ at ζ , and ψ be an analytic self map of \mathbb{D} satisfying $\psi(\mathbb{D}) \subset W(\varphi)$. Then*

$$C_\psi C_\varphi^* \equiv M_{g \circ \psi} C_{\sigma \circ \psi} \pmod{\mathcal{K}}$$

where $g(z) := \frac{z\sigma'(z)}{\sigma(z)}$.

In particular, $g \circ \psi$ is an $H^\infty(\mathbb{D})$ function and $\sigma \circ \psi$ is an analytic self-map of \mathbb{D} .

Proof. By Lemma 5.4, there exists a $K_1 \in \mathcal{K}$ such that

$$C_\psi C_\varphi^* = M_{h \circ \psi} C_{\sigma \circ \psi} S^* + K_1,$$

where h is defined by $h(z) = z\sigma'(z)$ and satisfies $h \circ \psi \in H^\infty(\mathbb{D})$, and $\sigma \circ \psi$ is an analytic self-map of \mathbb{D} . Note that since ψ maps \mathbb{D} into $W(\varphi)$, the map $\sigma \circ \psi$ is bounded away from zero on \mathbb{D} , and so $g \circ \psi$ is an $H^\infty(\mathbb{D})$ function. Recall that for any $f \in H^2$ and any nonzero $z \in \mathbb{D}$, we have that $(S^*f)(z) = \frac{f(z) - f(0)}{z}$, and so for any $f \in H^2$ and $z \in \mathbb{D}$ we have

$$\begin{aligned} (C_\psi C_\varphi^* f)(z) &= (M_{h \circ \psi} C_{\sigma \circ \psi} S^* f)(z) + (K_1 f)(z) \\ &= \psi(z) \sigma'(\psi(z)) \frac{f(\sigma(\psi(z))) - f(0)}{\sigma(\psi(z))} + (K_1 f)(z) \\ &= g(\psi(z)) f(\sigma(\psi(z))) - g(\psi(z)) f(0) + (K_1 f)(z). \end{aligned}$$

Thus $C_\psi C_\varphi^* = M_{g \circ \psi} C_{\sigma \circ \psi} - K_2 + K_1$, where K_2 is the rank one operator defined by $K_2: f \mapsto f(0) \cdot g \circ \psi$, and the proof is complete. \square

For a linear fractional map $\varphi(z) = \frac{az+b}{cz+d}$, we have that $\varphi_e(z)$ is invertible, and $\varphi_e^{-1}(z) = \sigma(z)$ is the Krein adjoint of φ . In [22], Kriete-MacCluer-Moorhouse developed the adjoint formula modulo \mathcal{K} for this case, which states

$$C_\varphi^* \equiv \frac{1}{|\varphi'(\zeta)|} C_\sigma \pmod{\mathcal{K}}.$$

This can easily be extended to $\tau \in \mathcal{S}$ with $F(\tau) = \{\zeta\}$, to produce a formula for C_τ^* provided τ has order of contact 2 with $\partial\mathbb{D}$ at ζ . We can now generalize this adjoint formula to higher orders of contact.

For what follows, recall that $f(\xi)$ denotes the non-tangential limit of f at ξ for a function f of \mathbb{D} and $\xi \in \partial\mathbb{D}$.

Proposition 5.6. *Let φ be a basic function with contact with $\partial\mathbb{D}$ at ζ and let ψ be a self-map of \mathbb{D} satisfying for some $\eta \in \partial\mathbb{D}$:*

- (i) $\psi(\mathbb{D}) \subset W(\varphi)$;
- (ii) ψ is analytic at η and $\psi(\eta) = \lambda$;
- (iii) $\psi^{-1}(\{\lambda\}) := \{\beta \in \partial\mathbb{D} : \psi(\beta) \text{ exists and is equal to } \lambda\} = \{\eta\}$.

Then we have that the map $\sigma \circ \psi$ is in the class \mathcal{S} with $F(\sigma \circ \psi) = \{\eta\}$ and that

$$C_\psi C_\varphi^* \equiv \frac{1}{|\varphi'(\zeta)|} C_{\sigma \circ \psi} \pmod{\mathcal{K}}.$$

Proof. By Lemma 5.5 we have that

$$(5.3) \quad C_\psi C_\varphi^* \equiv M_{g \circ \psi} C_{\sigma \circ \psi} \pmod{\mathcal{K}},$$

where $g(z) = \frac{z\sigma'(z)}{\sigma(z)}$, $g \circ \psi$ is an $H^\infty(\mathbb{D})$ function and $\sigma \circ \psi$ is an analytic self-map of \mathbb{D} . Note that it follows from (i) and (ii) and the definition of $W(\varphi)$ that the maps $\sigma \circ \psi$ and $g \circ \psi$ are both analytic at η . In order to apply Corollary 4.5 to $M_{g \circ \psi} C_{\sigma \circ \psi}$, it remains to show that $\sigma \circ \psi \in \mathcal{S}$.

Recall that $E(\sigma \circ \psi) = \bigcup_{|\alpha|=1} \overline{spt(\mu_\alpha^s)}$, where μ_α^s is the singular part of the Clark measure for $\sigma \circ \psi$ which is carried by

$$(\sigma \circ \psi)^{-1}(\{\alpha\}) = \{\beta \in \partial\mathbb{D} : (\sigma \circ \psi)(\beta) \text{ exists and is equal to } \alpha\}.$$

Suppose $(\sigma \circ \psi)(\beta) = \alpha$ for some $\alpha, \beta \in \partial\mathbb{D}$. Then applying φ_e we get that $\psi(\beta)$ exists and $\sigma(\psi(\beta)) = \alpha$. Recall that $\varphi_e^{-1}(\partial\mathbb{D}) = \{\zeta\}$, $\sigma(\lambda) = \zeta$ and σ is univalent on $V(\varphi)$. Thus $\sigma(\psi(\beta)) = \alpha$ implies that $\alpha = \zeta$ and $\psi(\beta) = \lambda$, and so by (iii) we have $\beta = \eta$. Therefore $spt(\mu_\alpha^s)$ is empty for $\alpha \neq \zeta$, and $spt(\mu_\zeta^s) \subset \{\eta\}$. We conclude that $E(\sigma \circ \psi) \subset \{\eta\}$ is finite and so $E(\sigma \circ \psi) = F(\sigma \circ \psi) = \{\eta\}$. Since $\sigma \circ \psi$ is analytic at η and does not map an arc of $\partial\mathbb{D}$ containing η into $\partial\mathbb{D}$, $\sigma \circ \psi$ has finite order of contact at η and a Taylor expansion to that order about η . Thus $\sigma \circ \psi$ is in the class \mathcal{S} .

Now applying Corollary 4.5 to $M_{g \circ \psi} C_{\sigma \circ \psi}$, Eq. (5.3) becomes

$$C_\psi C_\varphi^* \equiv (g \circ \psi)(\lambda) C_{\sigma \circ \psi} \pmod{\mathcal{K}}.$$

In order to complete the proof, we calculate $(g \circ \psi)(\eta) = g(\lambda) = \frac{\lambda}{\zeta\varphi'(\zeta)}$ and note that $|\varphi'(\zeta)| = \zeta\bar{\lambda}\varphi'(\zeta)$ by the Julia Carathéodory Theorem [15]. \square

5.3. Relationship Between φ and σ .

Let φ be a rational function with order of contact n with the unit circle at ζ , mapping ζ to λ , and suppose that σ is a branch of φ_e^{-1} defined on some neighborhood of λ and mapping λ to ζ . Then $\sigma \circ \varphi_e = id$ near ζ and $\varphi_e \circ \sigma = id$ near λ , so that $D_n(\sigma \circ \varphi_e, \zeta) = (\zeta, 1, 0, \dots, 0)$ and $D_n(\varphi_e \circ \sigma, \lambda) = (\lambda, 1, 0, \dots, 0)$. Although φ and σ are not inverse functions, we show that they are “almost inverse” in the sense that $D_{n-1}(\sigma \circ \varphi, \zeta) = (\zeta, 1, 0, \dots, 0)$, and $D_{n-1}(\varphi \circ \sigma, \lambda) = (\lambda, 1, 0, \dots, 0)$. However $D_n(\sigma \circ \varphi, \zeta)$ and $D_n(\varphi \circ \sigma, \lambda)$ are not generally equal, and the precise way that they can differ is one key to our main result.

Throughout this section, we use Theorem 2.1 (Faà di Bruno’s formula). We transfer \mathbb{D} to \mathbb{H} using the family of conformal maps $\tau_\alpha : \mathbb{D} \rightarrow \mathbb{H}$ for $\alpha \in \partial\mathbb{D}$, defined by $\tau_\alpha : z \mapsto i\frac{\alpha-z}{\alpha+z}$, and analyze the relationships of $\tilde{\varphi} = \tau_\lambda \circ \varphi \circ \tau_\zeta^{-1}$, $\tilde{\varphi}_e = \tau_\lambda \circ \varphi_e \circ \tau_\zeta^{-1}$ and $\tilde{\sigma} = \tau_\zeta \circ \sigma \circ \tau_\lambda^{-1}$. Note that $\tilde{\varphi}, \tilde{\sigma} \in \mathcal{P}_0$ and that $\tilde{\varphi}_e$ is the upper half-plane exterior map associated with $\tilde{\varphi}$, that is, $\tilde{\varphi}_e(z) = \overline{\tilde{\varphi}(\bar{z})}$.

Lemma 5.7. $D_{n-1}(\varphi, \zeta) = D_{n-1}(\varphi_e, \zeta)$ and $\varphi^{(n)}(\zeta) \neq \varphi_e^{(n)}(\zeta)$.

Proof. By the discussion in Section 2.1, it suffices to show that

$$D_{n-1}(\tilde{\varphi}, 0) = D_{n-1}(\tilde{\varphi}_e, 0) \quad \text{and} \quad \tilde{\varphi}^{(n)}(0) \neq \tilde{\varphi}_e^{(n)}(0).$$

Let $\tilde{\varphi}(z) = \sum_0^\infty a_k z^k$ be the Taylor expansion of $\tilde{\varphi}$ about 0. Then $\tilde{\varphi}_e(z) = \overline{\tilde{\varphi}(\bar{z})} = \sum_0^\infty \overline{a_k} z^k$, and so we have

$$\tilde{\varphi}^{(k)}(0) = k!a_k \quad \text{and} \quad \tilde{\varphi}_e^{(k)}(0) = k!\overline{a_k}$$

for all k . To complete the proof, recall that φ has order of contact n with the unit circle, and so $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$ and $\text{Im } a_n > 0$. \square

Proposition 5.8. *The n^{th} order data for $\sigma \circ \varphi$ and $\varphi \circ \sigma$ is given by*

$$D_n(\sigma \circ \varphi, \zeta) = \left(\zeta, 1, 0, \dots, 0, \frac{c}{\varphi'(\zeta)} \right)$$

$$D_n(\varphi \circ \sigma, \lambda) = \left(\lambda, 1, 0, \dots, 0, \frac{c}{\varphi'(\lambda)^n} \right),$$

where c is the non zero constant given by $c = \varphi^{(n)}(\zeta) - \varphi_e^{(n)}(\zeta)$.

Proof. By Lemma 5.7 $D_{n-1}(\varphi, \zeta) = D_{n-1}(\varphi_e, \zeta)$, and so by Theorem 2.1 (Faà di Bruno's formula), we get

$$D_{n-1}(\sigma \circ \varphi, \zeta) = D_{n-1}(\sigma \circ \varphi_e, \zeta) = (\zeta, 1, 0, \dots, 0),$$

$$D_{n-1}(\varphi \circ \sigma, \lambda) = D_{n-1}(\varphi_e \circ \sigma, \lambda) = (\lambda, 1, 0, \dots, 0).$$

We use Eq. (2.2) to write $(\sigma \circ \varphi)^{(n)}(\zeta)$ and $(\sigma \circ \varphi_e)^{(n)}(\zeta)$ as:

$$(\sigma \circ \varphi)^{(n)}(\zeta) = \sigma^{(n)}(\lambda) \varphi'(\zeta)^n + F(D_{n-1}(\sigma, \lambda), D_{n-1}(\varphi, \zeta)) + \sigma'(\lambda) \varphi^{(n)}(\zeta),$$

$$(\sigma \circ \varphi_e)^{(n)}(\zeta) = \sigma^{(n)}(\lambda) \varphi_e'(\zeta)^n + F(D_{n-1}(\sigma, \lambda), D_{n-1}(\varphi_e, \zeta)) + \sigma'(\lambda) \varphi_e^{(n)}(\zeta).$$

Now note that $(\sigma \circ \varphi_e)^{(n)}(\zeta) = 0$ and $D_{n-1}(\varphi, \zeta) = D_{n-1}(\varphi_e, \zeta)$, so that subtracting the above equations yields

$$(\sigma \circ \varphi)^{(n)}(\zeta) = \sigma'(\lambda) \cdot (\varphi^{(n)}(\zeta) - \varphi_e^{(n)}(\zeta)).$$

To complete the proof of the first statement, note that Lemma 5.7 implies that $\sigma'(\lambda) = 1/\varphi_e'(\zeta) = 1/\varphi'(\zeta)$ and that $c = \varphi^{(n)}(\zeta) - \varphi_e^{(n)}(\zeta)$ is non zero.

Similarly, we apply Eq. (2.2) to $(\varphi \circ \sigma)^{(n)}(\lambda)$ and $(\varphi_e \circ \sigma)^{(n)}(\lambda)$, to get

$$(\varphi \circ \sigma)^{(n)}(\zeta) = \varphi^{(n)}(\zeta) \sigma'(\lambda)^n + F(D_{n-1}(\varphi, \zeta), D_{n-1}(\sigma, \lambda)) + \varphi'(\zeta) \sigma^{(n)}(\lambda),$$

$$(\varphi_e \circ \sigma)^{(n)}(\zeta) = \varphi_e^{(n)}(\zeta) \sigma'(\lambda)^n + F(D_{n-1}(\varphi_e, \zeta), D_{n-1}(\sigma, \lambda)) + \varphi_e'(\zeta) \sigma^{(n)}(\lambda).$$

We subtract the above equations and substitute $1/\varphi'(\zeta)$ for $\sigma'(\lambda)$ to get

$$(\varphi \circ \sigma)^{(n)}(\zeta) = 1/\varphi'(\zeta)^n \cdot (\varphi^{(n)}(\zeta) - \varphi_e^{(n)}(\zeta)).$$

\square

As an additional application of Faà di Bruno's Formula, we calculate the order of contact of σ with $\partial\mathbb{D}$ at λ .

Proposition 5.9. *The map σ has order of contact n with $\partial\mathbb{D}$ at λ .*

Proof. Let $\tilde{\varphi}(z) = \sum_0^\infty a_k z^k$ and $\tilde{\sigma}(z) = \sum_0^\infty b_k z^k$ be the Taylor expansions of $\tilde{\varphi}$ and $\tilde{\sigma}$ about 0 respectively, and recall that $\tilde{\varphi}_e(z) = \overline{\tilde{\varphi}(\bar{z})} = \sum_0^\infty \overline{a_k} z^k$. Note that since φ has order of contact n with \mathbb{D} , we have that $a_0, \dots, a_{n-1} \in \mathbb{R}$ and $\text{Im } a_n > 0$.

It suffices to show that $b_0 = 0, b_1, \dots, b_{n-1} \in \mathbb{R}$ and $\text{Im } b_n > 0$. By induction on $k = 2, \dots, n-1$ and using Theorem 2.1 (Faà di Bruno's formula) for $(\tilde{\sigma} \circ \tilde{\varphi}_e)^{(k)}(0)$,

we see that b_0, \dots, b_k are real valued. To see that $\text{Im } b_n > 0$, we use Eq. (2.2) and write

$$0 = (\tilde{\sigma} \circ \tilde{\varphi}_e)^{(n)}(0) = \tilde{\sigma}^{(n)}(0) \tilde{\varphi}_e'(0)^n + F(D_{n-1}(\tilde{\sigma}, 0), D_{n-1}(\tilde{\varphi}_e, 0)) + \tilde{\sigma}'(0) \tilde{\varphi}_e^{(n)}(0).$$

Taking imaginary parts we get $\text{Im}(b_n) = \frac{b_1}{a_1^n} \text{Im}(a_n)$, and since $b_1 = 1/a_1 > 0$ we get that $\text{Im}(b_n) > 0$. \square

6. ESSENTIAL NORMALITY

We begin by characterizing essential normality for the basic case, then prove our main theorem characterizing essential normality for C_φ for all $\varphi \in \mathcal{S}$. Finally, we construct essentially normal composition operators which have arbitrary even order of contact with the unit circle at one point.

Definition 6.1. For $\epsilon > 0$ we define $\psi_{\lambda, \epsilon}$ to be the Riemann mapping from \mathbb{D} onto $\{|z - \lambda| < \epsilon\} \cap \mathbb{D}$, which fixes λ . Note that $\psi = \psi_{\lambda, \epsilon}$ extends continuously to $\partial\mathbb{D}$ and analytically across $\partial\mathbb{D}$ in a neighborhood of λ .

6.1. Essential Normality For Basic Composition Operators.

Proposition 6.2. *Let φ be a basic function with contact at ζ which fixes ζ . Let n be the order of contact of φ with $\partial\mathbb{D}$ at ζ , and let σ be the unique branch of φ_e^{-1} defined on a neighborhood of ζ which fixes ζ . There exists $\epsilon > 0$ such that for $\psi = \psi_{\zeta, \epsilon}$, the following conditions are equivalent.*

- (i) $C_\psi[C_\varphi^*, C_\varphi]$ is compact.
- (ii) $D_n(\sigma \circ \varphi, \zeta) = D_n(\varphi \circ \sigma, \zeta)$.
- (iii) $\varphi'(\zeta) = 1$.

Proof. (i) \iff (ii): For $W = W(\varphi)$, note that $\varphi^{-1}(W) \cap W$ is open and contains ζ , and choose $\epsilon > 0$ such that $\{|z - \zeta| \leq \epsilon\} \cap \mathbb{D}$ is contained in $\varphi^{-1}(W) \cap W$. Denote $\psi = \psi_{\zeta, \epsilon}$ and note that both ψ and $\varphi \circ \psi$ map \mathbb{D} into W , are analytic at ζ and fix ζ and satisfy $\psi^{-1}(\{\zeta\}) = (\varphi \circ \psi)^{-1}(\{\zeta\}) = \{\zeta\}$, and so by Proposition 5.6 we have

$$C_\psi C_\varphi^* \equiv \frac{1}{|\varphi'(\zeta)|} C_{\sigma \circ \psi} \pmod{\mathcal{K}}, \quad C_{\varphi \circ \psi} C_\varphi^* \equiv \frac{1}{|\varphi'(\zeta)|} C_{\sigma \circ \varphi \circ \psi} \pmod{\mathcal{K}}.$$

Thus, we can express $C_\psi[C_\varphi^*, C_\varphi]$ in the Calkin algebra by

$$C_\psi[C_\varphi^*, C_\varphi] \equiv C_\psi C_\varphi^* C_\varphi - C_{\varphi \circ \psi} C_\varphi^* \equiv \frac{1}{|\varphi'(\zeta)|} (C_{\varphi \circ \sigma \circ \psi} - C_{\sigma \circ \varphi \circ \psi}) \pmod{\mathcal{K}}.$$

Note that ψ maps an arc of $\partial\mathbb{D}$ containing ζ into $\partial\mathbb{D}$, and recall that by Proposition 5.9 σ has order of contact n at ζ . Thus, by Proposition 2.4 both $\sigma \circ \varphi \circ \psi$ and $\varphi \circ \sigma \circ \psi$ have order of contact n at ζ . Note also that $\sigma \circ \varphi \circ \psi$ and $\varphi \circ \sigma \circ \psi$ are in the class \mathcal{S} with $F(\sigma \circ \varphi \circ \psi) = F(\varphi \circ \sigma \circ \psi) = \{\zeta\}$. Therefore, by Theorem 4.1, $C_{\sigma \circ \varphi \circ \psi} - C_{\varphi \circ \sigma \circ \psi}$ is compact if and only if $D_n(\sigma \circ \varphi \circ \psi, \zeta) = D_n(\varphi \circ \sigma \circ \psi, \zeta)$. Since ψ is invertible in a neighborhood of ζ and as a consequence of Theorem 2.1 (Faà di Bruno's formula), we get that $D_n(\sigma \circ \varphi \circ \psi, \zeta) = D_n(\varphi \circ \sigma \circ \psi, \zeta)$ if and only if $D_n(\sigma \circ \varphi, \zeta) = D_n(\varphi \circ \sigma, \zeta)$.

(ii) \iff (iii): By Proposition 5.8, the n^{th} order data for $\sigma \circ \varphi$ and $\varphi \circ \sigma$ is equal if and only if $\varphi'(\zeta) = \varphi'(\zeta)^n$. Note that $\varphi'(\zeta) > 0$ since φ fixes ζ , and so $\varphi'(\zeta) = \varphi'(\zeta)^n$ if and only if $\varphi'(\zeta) = 1$. \square

If C_ψ were bounded below, compactness of $C_\psi A$ would imply compactness of A for any operator A . We prove a weaker result using a technique from [12, section 4]. We denote the characteristic function on $\partial\mathbb{D}$ for the arc $\Gamma_\delta = \{|z - \zeta| < \delta\} \cap \partial\mathbb{D}$ by χ_δ .

Lemma 6.3. *Let $\epsilon > 0$ and $\psi = \psi_{\zeta, \epsilon}$. Then for any $\chi = \chi_\delta$ with $0 < \delta < \epsilon$, and any operator A we have*

$$C_\psi A \text{ is compact} \implies T_\chi A \text{ is compact.}$$

Proof. Suppose that $C_\psi A$ is compact. It suffices to show that the H^2 to L^2 operator $M_\chi A$ is compact. Let t_1, t_2 be such that e^{it} parametrizes the curve Γ_δ for $t_1 \leq t \leq t_2$, and let α_1, α_2 be such that $\psi(e^{it})$ parametrizes the curve Γ_δ for $\alpha_1 \leq t \leq \alpha_2$. Then

$$\begin{aligned} \|M_\chi f\|_{L^2}^2 &= \int_{t_1}^{t_2} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \int_{\alpha_1}^{\alpha_2} |f(\psi(e^{it}))|^2 |\psi'(e^{it})| \frac{dt}{2\pi} \\ &\leq m \int_{\alpha_1}^{\alpha_2} |f(\psi(e^{it}))|^2 \frac{dt}{2\pi} \leq m \|C_\psi f\|_{L^2}^2, \end{aligned}$$

where $m = \max\{|\psi'(e^{it})| : \alpha_1 \leq t \leq \alpha_2\}$ is finite since ψ can be analytically extended to a neighborhood of Γ .

Note that $C_\psi A$ is compact from H^2 to H^2 and so it is compact from H^2 to L^2 , and that for all $f \in H^2$ we have that $\|M_\chi A f\|_{L^2} \leq \sqrt{m} \|C_\psi A f\|_{L^2}$. Thus $M_\chi A$ is compact from H^2 to L^2 . \square

Proposition 6.4. *Suppose φ is a basic function with contact at ζ which fixes ζ . Then C_φ is essentially normal if and only if $\varphi'(\zeta) = 1$.*

Proof. By Proposition 6.2, there exists $\epsilon > 0$ such that for $\psi = \psi_{\zeta, \epsilon}$, we have $C_\psi[C_\varphi^*, C_\varphi]$ is compact if and only if $\varphi'(\zeta) = 1$. Thus, it suffices to show that compactness of $C_\psi[C_\varphi^*, C_\varphi]$ implies compactness of $[C_\varphi^*, C_\varphi]$.

Suppose that $C_\psi[C_\varphi^*, C_\varphi]$ is compact. Then by Lemma 6.3 with $\delta = \epsilon/2$ and $\chi = \chi_\delta$, we get that $T_\chi[C_\varphi^*, C_\varphi]$ is compact. Now since $\varphi \in \mathcal{S}$ with $F(\varphi) = \{\zeta\}$ and χ is continuously differentiable at ζ , using Corollary 4.5 and Proposition 4.7 we get

$$T_\chi[C_\varphi^*, C_\varphi] \equiv \chi(\varphi(\zeta))C_\varphi^*C_\varphi - \chi(\zeta)C_\varphi C_\varphi^* \equiv [C_\varphi^*, C_\varphi] \pmod{\mathcal{K}},$$

and so $[C_\varphi^*, C_\varphi]$ is compact as well. \square

6.2. General Essential Normality.

In this section we prove our main theorem identifying the essentially normal composition operators induced by a general function in the class \mathcal{S} . The first statement in the following lemma is due to Clifford and Zheng [13] in the case where φ_1 and φ_2 are linear fractional maps.

Lemma 6.5. *Let φ_1 and φ_2 be basic functions with contact at ζ_1 and ζ_2 respectively, and denote $\lambda_1 = \varphi_1(\zeta_1)$ and $\lambda_2 = \varphi_2(\zeta_2)$. Then the following hold.*

- (i) *If $\zeta_1 \neq \zeta_2$ then $C_{\varphi_1}^* C_{\varphi_2} \equiv 0 \pmod{\mathcal{K}}$.*
- (ii) *If $\lambda_1 \neq \lambda_2$ then $C_{\varphi_2} C_{\varphi_1}^* \equiv 0 \pmod{\mathcal{K}}$.*

Proof. For the first part, suppose that $\zeta_1 \neq \zeta_2$ and note that $b(z) = \frac{z - \zeta_1}{\zeta_2 - \zeta_1}$ is continuous at ζ_1 and ζ_2 and satisfies $b(\zeta_1) = 0$ and $b(\zeta_2) = 1$. Then by Corollary 4.5, $T_b C_{\varphi_2} \equiv C_{\varphi_2}$ and $T_b C_{\varphi_1} \equiv 0 \pmod{\mathcal{K}}$, and so,

$$C_{\varphi_1}^* C_{\varphi_2} \equiv C_{\varphi_1}^* (T_b C_{\varphi_2}) \equiv C_{\varphi_1}^* T_b^* C_{\varphi_2} \equiv (T_b C_{\varphi_1})^* C_{\varphi_2} \equiv 0 \pmod{\mathcal{K}}.$$

For the second part, suppose $\lambda_1 \neq \lambda_2$ and note that $c(z) = \frac{z-\lambda_2}{\lambda_1-\lambda_2}$ is continuously differentiable in neighborhoods of λ_1 and λ_2 and satisfies $c(\lambda_1) = 1$ and $c(\lambda_2) = 0$. Then by Proposition 4.7, $T_c C_{\varphi_1}^* \equiv C_{\varphi_1}^*$ and $T_{\bar{c}} C_{\varphi_2}^* \equiv 0 \pmod{\mathcal{K}}$, and so

$$C_{\varphi_2} C_{\varphi_1}^* \equiv C_{\varphi_2} (T_c C_{\varphi_1}^*) \equiv (C_{\varphi_2}^*)^* (T_{\bar{c}})^* C_{\varphi_1}^* \equiv (T_{\bar{c}} C_{\varphi_2}^*)^* C_{\varphi_1}^* \equiv 0 \pmod{\mathcal{K}}.$$

□

Proposition 6.6. *Let φ in \mathcal{S} be such that C_φ is essentially normal. Then φ permutes the elements of $F(\varphi)$. Furthermore, if $\varphi(\zeta) = \zeta'$ for some $\zeta, \zeta' \in F(\varphi)$, then φ has equal order of contact, say n , with $\partial\mathbb{D}$ at ζ and ζ' , and we have*

$$\frac{\varphi'(\zeta)^n}{\varphi'(\zeta')} = \frac{c}{c'},$$

where c and c' are non zero constants uniquely determined by $D_n(\varphi, \zeta)$ and $D_n(\varphi, \zeta')$ respectively.

Proof. Let $F(\varphi) = \{\zeta_1 = \zeta, \dots, \zeta_r\}$ and let $\varphi_1, \dots, \varphi_r$ be the basic functions with contact of order n_1, \dots, n_r with $\partial\mathbb{D}$ at ζ_1, \dots, ζ_r respectively, guaranteed by Theorem 4.4, so that

$$C_\varphi \equiv C_{\varphi_1} + \dots + C_{\varphi_r} \pmod{\mathcal{K}}.$$

For each j , we denote $\lambda_j = \varphi(\zeta_j)$, and let σ_j be the unique branch of $(\varphi_j)_e^{-1}$ defined in some neighborhood of λ_j which maps λ_j to ζ_j .

Let b be a polynomial such that $b(\zeta_1) = 1$ and b is 0 at all the points in $\{\zeta_2, \dots, \zeta_r, \lambda_1, \dots, \lambda_r\} \setminus \{\zeta_1\}$. Then by Lemma 6.5, Corollary 4.5 and Proposition 4.7 we get that

$$\begin{aligned} M_b[C_\varphi, C_\varphi^*] &\equiv \sum_{j,k=1,\dots,r} M_b C_{\varphi_k} C_{\varphi_j}^* - M_b C_{\varphi_j}^* C_{\varphi_k} \\ (6.1) \quad &\equiv \sum_{\substack{k=1,\dots,r \\ \lambda_k=\lambda_1}} C_{\varphi_1} C_{\varphi_k}^* - \sum_{\substack{j=1,\dots,r \\ \lambda_j=\zeta_1}} C_{\varphi_j}^* C_{\varphi_j} \pmod{\mathcal{K}}. \end{aligned}$$

We define a neighborhood W of ζ_1 by

$$W = \left(\bigcap_{\lambda_j=\zeta_1} W_j \right) \cap \left(\bigcap_{\lambda_k=\lambda_1} \varphi_1^{-1}(W_k) \right),$$

where $W_j = W(\varphi_j)$ denotes the neighborhood of λ_j used in Proposition 5.6, and let $\psi = \psi_{\zeta_1, \epsilon}$ with $\epsilon > 0$ such that $\{|z - \zeta_1| \leq \epsilon\} \cap \mathbb{D}$ is contained in W . Note that for each j such that $\lambda_j = \zeta_1$, the map ψ satisfies

- $\psi(\mathbb{D}) \subset W_j$;
- ψ is analytic at ζ_1 and $\psi(\zeta_1) = \zeta_1$;
- $\psi^{-1}(\{\zeta_1\}) = \{\zeta_1\}$.

Similarly, for each k such that $\lambda_k = \lambda_1$, the map $\varphi_1 \circ \psi$ satisfies

- $(\varphi_1 \circ \psi)(\mathbb{D}) \subset \varphi_1(W) \subset W_k$;
- $\varphi_1 \circ \psi$ is analytic at ζ_1 and $(\varphi_1 \circ \psi)(\zeta_1) = \lambda_k$;
- $(\varphi_1 \circ \psi)^{-1}(\{\lambda_k\}) = \{\zeta_1\}$.

Using Eq. (6.1) and applying Proposition 5.6 multiple times, we get

$$\begin{aligned} C_\psi M_b[C_\varphi, C_\varphi^*] &\equiv \sum_{\substack{k=1\dots r \\ \lambda_k=\lambda_1}} C_{\varphi_1 \circ \psi} C_{\varphi_k}^* - \sum_{\substack{j=1\dots r \\ \lambda_j=\zeta_1}} C_\psi C_{\varphi_j}^* C_{\varphi_j} \\ &\equiv \sum_{\substack{k=1\dots r \\ \lambda_k=\lambda_1}} \frac{1}{|\varphi'(\zeta_k)|} C_{\sigma_k \circ \varphi_1 \circ \psi} - \sum_{\substack{j=1\dots r \\ \lambda_j=\zeta_1}} \frac{1}{|\varphi'(\zeta_j)|} C_{\varphi_j \circ \sigma_j \circ \psi} \pmod{\mathcal{K}}, \end{aligned}$$

where all the inducing maps $\sigma_k \circ \varphi_1 \circ \psi$ and $\varphi_j \circ \sigma_j \circ \psi$ above are in the class \mathcal{S} with $F(\sigma_k \circ \varphi_1 \circ \psi) = F(\varphi_j \circ \sigma_j \circ \psi) = \{\zeta_1\}$. Note that the maps $\sigma_k \circ \varphi_1 \circ \psi$ above all map ζ_1 to ζ_k and the maps $\varphi_j \circ \sigma_j \circ \psi$ above all fix ζ_1 .

Recalling that C_φ is essentially normal, we see that

$$(6.2) \quad \sum_{\substack{k=1\dots r \\ \lambda_k=\lambda_1}} \frac{1}{|\varphi'(\zeta_k)|} C_{\sigma_k \circ \varphi_1 \circ \psi} - \sum_{\substack{j=1\dots r \\ \lambda_j=\zeta_1}} \frac{1}{|\varphi'(\zeta_j)|} C_{\varphi_j \circ \sigma_j \circ \psi}$$

is a compact linear combination of composition operators. Now Theorem 4.1 combined with the properties of the inducing maps in Eq. (6.2) imply that $\lambda_k \neq \lambda_1$ for $k \neq 1$. By symmetry we conclude that $\lambda_1, \dots, \lambda_r$ are all distinct. Thus the first sum in Eq. (6.2) consists of exactly one term, and the second sum consists of at most one term. Again using Theorem 4.1 we see that the second sum must also consist of exactly one term and so there is a unique j_1 such that $\lambda_{j_1} = \varphi(\zeta_{j_1}) = \zeta_1$. By symmetry, for any $k \in \{1, \dots, r\}$ there exists a unique j_k such that $\lambda_{j_k} = \varphi(\zeta_{j_k}) = \zeta_k$. Thus, φ acts as a one-to-one map of $F(\varphi)$ onto itself. This proves the first statement.

Furthermore, Eq. (6.2) becomes $\frac{1}{|\varphi'(\zeta_1)|} C_{\sigma_1 \circ \varphi_1 \circ \psi} - \frac{1}{|\varphi'(\zeta_{j_1})|} C_{\varphi_{j_1} \circ \sigma_{j_1} \circ \psi}$, and so by Theorem 4.1, we have that $\sigma_1 \circ \varphi_1 \circ \psi$ and $\varphi_{j_1} \circ \sigma_{j_1} \circ \psi$ have equal order of contact, say n , at ζ_1 and $D_n(\sigma_1 \circ \varphi_1 \circ \psi, \zeta_1) = D_n(\varphi_{j_1} \circ \sigma_{j_1} \circ \psi, \zeta_1 = \lambda_{j_1})$. We use Proposition 2.4 combined with Proposition 5.9 and the properties of ψ to conclude that φ has order of contact n with $\partial\mathbb{D}$ at both ζ_1 and ζ_{j_1} . We use Theorem 2.1 (Faà di Bruno's formula) combined with invertibility of ψ in a neighborhood of ζ to conclude that $D_n(\sigma_1 \circ \varphi_1, \zeta_1) = D_n(\varphi_{j_1} \circ \sigma_{j_1}, \lambda_{j_1})$, and in particular $(\sigma_1 \circ \varphi_1)^{(n)}(\zeta_1) = (\varphi_{j_1} \circ \sigma_{j_1})^{(n)}(\lambda_{j_1})$. Now by Proposition 5.8 we get that $\frac{c_1}{\varphi'_1(\zeta_1)} = \frac{c_{j_1}}{\varphi'_{j_1}(\zeta_{j_1})^n}$, where $c_j = \varphi_j^{(n_j)}(\zeta_j) - (\varphi_j)_e^{(n_j)}(\zeta_j)$ is a non zero constant determined by $D_{n_j}(\varphi, \zeta_j)$. By symmetry, the proof is complete. \square

We now have the tools to prove our main theorem identifying the non trivially essentially normal composition operators induced by maps in \mathcal{S} .

Theorem 6.7. *Let φ be in \mathcal{S} . Then C_φ is non trivially essentially normal if and only if $F(\varphi) = \{\zeta\}$ for some $\zeta \in \partial\mathbb{D}$, φ fixes ζ , and $\varphi'(\zeta) = 1$.*

Proof. First note that for the case where $F(\varphi)$ is empty we have that $E(\varphi)$ is empty. Thus, as noted in Section 2.3, C_φ is compact and so trivially essentially normal in this case. For the case that $F(\varphi)$ contains one point, i.e. $|F(\varphi)| = 1$, we see that the statement is true by using Theorem 4.4 to reduce to the basic case considered in Proposition 6.4.

Now suppose that $|F(\varphi)| > 1$ and, in order to obtain a contradiction, suppose that C_φ is essentially normal. By Proposition 6.6 φ permutes the points in $F(\varphi)$ and so decomposes $F(\varphi)$ into disjoint cycles. If $\zeta \in F(\varphi)$ is a fixed point, we have

by Proposition 6.6 that $\varphi'(\zeta)^{n-1} = 1$ and since $\varphi(\zeta) > 0$ in this case, we get that ζ is the unique Denjoy-Wolff point of φ . Thus there is at most one φ -cycle of length 1 in $F(\varphi)$. Since $|F(\varphi)| > 1$, we must therefore have a φ -cycle $(\zeta_1, \dots, \zeta_k)$ in $F(\varphi)$ for some $k > 1$.

By Proposition 6.6 we get that φ has equal order of contact n with $\partial\mathbb{D}$ at ζ_1, \dots, ζ_k and the following equations hold

$$\frac{\varphi'(\zeta_1)^n}{\varphi'(\zeta_2)} = \frac{c_1}{c_2}, \quad \dots \quad \frac{\varphi'(\zeta_{k-1})^n}{\varphi'(\zeta_k)} = \frac{c_{k-1}}{c_k}, \quad \frac{\varphi'(\zeta_k)^n}{\varphi'(\zeta_1)} = \frac{c_k}{c_1},$$

where c_j is a non zero constant uniquely determined by $D_n(\varphi, \zeta_j)$. Taking the product of the above equations yields

$$\frac{\varphi'(\zeta_1)^n \cdot \varphi'(\zeta_2)^n \cdot \dots \cdot \varphi'(\zeta_k)^n}{\varphi'(\zeta_2) \cdot \dots \cdot \varphi'(\zeta_k) \cdot \varphi'(\zeta_1)} = \frac{c_1 \cdot c_2 \cdot \dots \cdot c_k}{c_2 \cdot \dots \cdot c_k \cdot c_1},$$

and so we get that

$$(6.3) \quad (\varphi'(\zeta_1) \cdot \dots \cdot \varphi'(\zeta_k))^{n-1} = 1.$$

Since $(\zeta_1, \dots, \zeta_k)$ is a cycle under φ , for each $j = 1, \dots, k$, the k -fold iterate $\varphi_{(k)}$ of φ fixes ζ_j and satisfies $\varphi'_{(k)}(\zeta_j) = \varphi'(\zeta_1) \cdot \dots \cdot \varphi'(\zeta_k)$ by the chain rule. It follows that $\varphi'_{(k)}(\zeta_j) > 0$ and by Eq. (6.3) $(\varphi'_{(k)}(\zeta_j))^{n-1} = 1$. Thus $\varphi'_{(k)}(\zeta_j) = 1$ and ζ_1, \dots, ζ_k all qualify as the unique Denjoy-Wolff point of $\varphi_{(k)}$, and we have reached a contradiction. \square

Corollary 6.8. *Let φ be a self-map of \mathbb{D} which extends analytically to a neighborhood of $\overline{\mathbb{D}}$. Then C_φ is non-trivially essentially normal if and only if there exists $\zeta \in \partial\mathbb{D}$ such that φ fixes ζ , $\varphi'(\zeta) = 1$, and φ maps $\partial\mathbb{D} \setminus \{\zeta\}$ into \mathbb{D} .*

Proof. We can assume that φ is not linear fractional. If φ is a finite Blaschke product of degree at least 2, then C_φ is not essentially normal by [16]. Otherwise, by Proposition 2.6, $\varphi \in \mathcal{S}$. If $F(\varphi) = E(\varphi)$ is empty, then C_φ is compact (as noted in Section 2.3) and so trivially essentially normal. Otherwise, the claim follows from Theorem 6.7. \square

6.3. Construction of Essentially Normal Composition Operators.

We combine the criterion for essential normality given in Theorem 6.7 with several results from Section 3 to construct essentially normal composition operators which have arbitrary even order of contact with $\partial\mathbb{D}$ at a point ζ .

Recall from Section 3 that for a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we denote the corresponding linear fractional transformation by $L[A]$, so that $L[A]h = \frac{a_{11}h + a_{12}}{a_{21}h + a_{22}}$, and use this notation to express the augmentation f of g by a_0, a_1 by $f(z) = L[A(a_0, a_1)(z)]g(z)$, where $A(a_0, a_1)(z)$ is defined by

$$A(a_0, a_1)(z) = \begin{bmatrix} a_0 a_1 z & -a_0 - a_1 z \\ a_1 z & -1 \end{bmatrix}.$$

Additionally, recall from Section 2.2 that for $\alpha \in \partial\mathbb{D}$, the map $\tau_\alpha: z \mapsto i \frac{\alpha - z}{\alpha + z}$ is a conformal map which we use to transfer \mathbb{D} to \mathbb{H} .

Theorem 6.9. *Let φ be a rational self-map of \mathbb{D} which extends analytically to a neighborhood of $\overline{\mathbb{D}}$. Then C_φ is non-trivially essentially normal if and only if φ is of the form $\varphi = \tau_\zeta^{-1} \circ f \circ \tau_\zeta$ for some $\zeta \in \partial\mathbb{D}$ with*

$$f(z) = L[A(0, 1)(z)A(s_1, t_1)(z) \cdots A(s_{m-1}, t_{m-1})(z)]w(z),$$

for some $n = 2m$, $s_1, \dots, s_{m-1} \in \mathbb{R}$, $t_1, \dots, t_{m-1} > 0$ and w a rational self-map of \mathbb{H} which maps $\widehat{\mathbb{R}}$ into \mathbb{H} .

Furthermore, this representation is unique, $\zeta \in \partial\mathbb{D}$ is the fixed point of φ and $n = 2m$ is the order of contact of φ with $\partial\mathbb{D}$ at ζ .

Proof. For the first direction, suppose that C_φ is non-trivially essentially normal. By Corollary 6.8, there exists $\zeta \in \partial\mathbb{D}$ such that φ fixes ζ and $\varphi'(\zeta) = 1$, and φ maps $\partial\mathbb{D} \setminus \{\zeta\}$ into \mathbb{D} . Note that for $\zeta \neq \eta \in \partial\mathbb{D}$, $f = \tau_\eta \circ \varphi \circ \tau_\eta^{-1}$ does not fix 0 and so is not of the above form. The properties of φ ensure that $f = \tau_\zeta \circ \varphi \circ \tau_\zeta^{-1}$ is a rational self-map of \mathbb{H} which fixes 0, satisfies $f'(0) = 1$ and maps $\widehat{\mathbb{R}} \setminus \{0\}$ into \mathbb{H} . Note that since φ is analytic in a neighborhood of ζ , φ has order of contact $n = 2m$ with $\partial\mathbb{D}$ at ζ , and so f has order of contact $n = 2m$ with \mathbb{R} at 0, for some positive integer m . By Proposition 3.6 we see that f has a unique representation in the form

$$f(z) = L[A(s_0, t_0)(z) \cdots A(s_{m-1}, t_{m-1})(z)]w(z),$$

where the self-map w of \mathbb{H} satisfies $w(0) \in \mathbb{H}$ and $s_1, \dots, s_{m-1} \in \mathbb{R}$, $t_1, \dots, t_{m-1} > 0$ are uniquely determined by f . Note that $f(0) = 0$ and $f'(0) = 1$, so that $s_0 = 0$ and $t_0 = 1$. Since w can be obtained from f by taking m reductions, and by the observation in Section 3.2, w is a rational function mapping $\widehat{\mathbb{R}} \setminus \{0\}$ into \mathbb{H} .

For the second direction, let φ be of the above form. By Proposition 3.6, f has order of contact n with \mathbb{R} at $z = 0$, and so φ has order of contact n with $\partial\mathbb{D}$ at ζ . Note that f is obtained by taking m augmentations of w and so by the observation in Section 3.2 and our assumptions on w , we get that f is rational and maps $\widehat{\mathbb{R}} \setminus \{0\}$ into \mathbb{H} . Since the last augmentation performed to obtain f has parameters 0 and 1, f satisfies $f(0) = 0$ and $f'(0) = 1$. The properties of f ensure that $\varphi = \tau_\zeta^{-1} \circ f \circ \tau_\zeta$ is a rational self-map of \mathbb{D} which extends analytically to a neighborhood of $\overline{\mathbb{D}}$ and has order of contact n with $\partial\mathbb{D}$ at ζ , fixes ζ and satisfies $\varphi'(\zeta) = 1$. By Corollary 6.8, the operator C_φ is non-trivially essentially normal. \square

Note that if w is chosen to be a constant function in the formula above, then φ is a degree m rational self-map of \mathbb{D} with order of contact $n = 2m$ with $\partial\mathbb{D}$ at ζ which induces an essentially normal composition operator.

Example. Let $\zeta \in \partial\mathbb{D}$, $s \in \mathbb{R}$, $t > 0$ and $w \in \mathbb{H}$ and define $f(s, t, w)$ by

$$\begin{aligned} f(s, t, w)(z) &= L[A(0, 1)(z)A(s, t)(z)]w \\ &= L \left[\begin{bmatrix} 0 & -z \\ z & -1 \end{bmatrix} \begin{bmatrix} sz & -s - tz \\ tz & -1 \end{bmatrix} \right] w = \frac{-tz^2w + z}{(stz^2 - tz)w - sz - tz^2 + 1}. \end{aligned}$$

Then for any $\zeta \in \partial\mathbb{D}$, $\varphi(\zeta, s, t, w) = \tau_\zeta^{-1} \circ f(s, t, w) \circ \tau_\zeta$ has order of contact 4 with $\partial\mathbb{D}$ at ζ and induces a non trivially essentially normal composition operator. As a

concrete example, we calculate $\varphi(1, 0, 1, i)(z) = \frac{z^2 + 2z + 1}{z^2 - 2z + 5}$.

REFERENCES

- [1] J. Agler, Z. A. Lykova, and N. J. Young. The boundary Carathéodory-Fejér interpolation problem. *J. Math. Anal. Appl.*, 382(2):645–662, 2011.
- [2] J. Agler, Z. A. Lykova, and N. J. Young. Pseudo-Taylor expansions and the Carathéodory-Fejér problem. *J. Math. Anal. Appl.*, 386(1):308–318, 2012.
- [3] J. Agler and N. J. Young. Boundary Nevanlinna-Pick interpolation via reduction and augmentation. *Math. Z.*, 268(3-4):791–817, 2011.
- [4] J. A. Ball, I. Gohberg, and L. Rodman. *Interpolation of rational matrix functions*, volume 45 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1990.
- [5] V. Bolotnikov. On higher order boundary derivatives of an analytic self-map of the unit disk. *Journal of Approximation Theory*, 163(4):568 – 589, 2011.
- [6] V. Bolotnikov and S. P. Cameron. The nevanlinna-pick problem on the closed unit disk: Minimal norm rational solutions of low degree. *Journal of Computational and Applied Mathematics*, 236(13):3123 – 3136, 2012.
- [7] P. S. Bourdon, D. Levi, S. K. Narayan, and J. H. Shapiro. Which linear-fractional composition operators are essentially normal? *J. Math. Anal. Appl.*, 280(1):30–53, 2003.
- [8] P. S. Bourdon and J. H. Shapiro. Adjoints of rationally induced composition operators. *J. Funct. Anal.*, 255(8):1995–2012, 2008.
- [9] C. Carathéodory and L. Fejér. ber den zusammenhang der extremen von harmonischen funktionen mit ihren koeffizienten und ber den picard-landauschen satz. *Rendiconti del Circolo Matematico di Palermo*, 32(1):218–239, 1911.
- [10] J. A. Cima and A. L. Matheson. Essential norms of composition operators and Aleksandrov measures. *Pacific J. Math.*, 179(1):59–64, 1997.
- [11] J. A. Cima, A. L. Matheson, and W. T. Ross. *The Cauchy transform*, volume 125 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006.
- [12] J. A. Cima, J. Thomson, and W. Wogen. On some properties of composition operators. *Indiana Univ. Math. J.*, 24:215–220, 1974/75.
- [13] J. H. Clifford and D. Zheng. Composition operators on the Hardy space. *Indiana Univ. Math. J.*, 48(4):1585–1616, 1999.
- [14] C. C. Cowen and E. A. Gallardo-Gutiérrez. A new class of operators and a description of adjoints of composition operators. *J. Funct. Anal.*, 238(2):447–462, 2006.
- [15] C. C. Cowen and B. D. MacCluer. *Composition operators on spaces of analytic functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.
- [16] H. Hamada. Quotient algebras of toeplitz-composition c^* -algebras for finite blaschke products. *Complex Analysis and Operator Theory*, 8(4):843–862, 2014.
- [17] C. Hammond, J. Moorhouse, and M. E. Robbins. Adjoints of composition operators with rational symbol. *J. Math. Anal. Appl.*, 341(1):626–639, 2008.
- [18] K. Hoffman. *Banach spaces of analytic functions*. Prentice-Hall Series in Modern Analysis. Prentice-Hall Inc., Englewood Cliffs, N. J., 1962.
- [19] W. P. Johnson. The curious history of Faà di Bruno’s formula. *Amer. Math. Monthly*, 109(3):217–234, 2002.
- [20] G. Julia. Extension nouvelle d’un lemme de Schwarz. *Acta Math.*, 42(1):349–355, 1920.
- [21] T. Kriete and J. Moorhouse. Linear relations in the Calkin algebra for composition operators. *Trans. Amer. Math. Soc.*, 359(6):2915–2944 (electronic), 2007.
- [22] T. L. Kriete, B. D. MacCluer, and J. L. Moorhouse. Toeplitz-composition C^* -algebras. *J. Operator Theory*, 58(1):135–156, 2007.
- [23] N. K. Nikolski. *Operators, functions, and systems: an easy reading. Vol. 1*, volume 92 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002. Hardy, Hankel, and Toeplitz, Translated from the French by Andreas Hartmann.
- [24] D. Sarason. Composition operators as integral operators. In *Analysis and partial differential equations*, volume 122 of *Lecture Notes in Pure and Appl. Math.*, pages 545–565. Dekker, New York, 1990.
- [25] D. Sarason. *Sub-Hardy Hilbert spaces in the unit disk*. University of Arkansas Lecture Notes in the Mathematical Sciences, 10. John Wiley & Sons Inc., New York, 1994. A Wiley-Interscience Publication.
- [26] H. J. Schwartz. *COMPOSITION OPERATORS ON $H(\mathcal{P})$* . ProQuest LLC, Ann Arbor, MI, 1969. Thesis (Ph.D.)—The University of Toledo.

- [27] J. H. Shapiro. The essential norm of a composition operator. *Ann. of Math. (2)*, 125(2):375–404, 1987.
- [28] J. H. Shapiro and C. Sundberg. Compact composition operators on L^1 . *Proc. Amer. Math. Soc.*, 108(2):443–449, 1990.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, PO Box 400137, CHARLOTTESVILLE,
VA 22904-4137, UNITED STATES

E-mail address: `md3ny@virginia.edu`